PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 3, December 1979

INTEGRALS OF CERTAIN UNIVALENT FUNCTIONS

RAM SINGH AND SUNDER SINGH

ABSTRACT. In this paper we study integrals of certain univalent functions in the unit disc $E = \{z: |z| < 1\}$ and extend some well-known results of Libera.

Introduction. Let A denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are regular in the unit disc $E = \{z: |z| < 1\}$. We denote by S the subclass of univalent functions in A and by C, S^* and K the subclasses of S whose members are close-to-convex, starlike (with respect to the origin) and convex in E, respectively. A function $f \in A$ is said to be starlike of order α, $\alpha < 1$, in E if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \qquad z \in E. \tag{1}$$

Similarly, $f \in A$ is said to be convex of order α, $\alpha < 1$, in E if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad z \in E. \tag{2}$$

We shall denote by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of A whose members satisfy (1) and (2), respectively. It is known that for $0 < \alpha < 1$, $S^*(\alpha) \subset S^*$, $K(\alpha) \subset K$ and that $S^*(0) \equiv S^*$, $K(0) \equiv K$.

Following Ruscheweyh [3] we denote by K_n, $n \in N_0 = \{0, 1, \dots\}$, the subclass of A whose members satisfy the condition

$$\operatorname{Re}\left\{\frac{(z^n f(z))^{(n+1)}}{(z^{n-1} f(z))^{(n)}}\right\} > \frac{n+1}{2}, \qquad z \in E. \tag{3}$$

It is readily seen that $K_0 \equiv S^*(1/2)$ and $K_1 \equiv K$.

As observed by Ruscheweyh, $f \in K_n$ if and only if

$$\operatorname{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \frac{1}{2} \qquad (z \in E), \tag{4}$$

Received by the editors June 13, 1978.

AMS (MOS) subject classifications (1970). Primary 30A32, 30A34.

Key words and phrases. Univalent, starlike and convex functions, convolution/Hadamard product.

0002-9939/79/0000-0558/$02.25

where

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z).$$

(Here '$*$' stands for the Hadamard product/convolution of two regular functions.)

In [3] Ruscheweyh proved that for each $n \in N_0$, $K_{n+1} \subset K_n$. Since $K_0 \equiv S^*(1/2)$, Ruscheweyh's result implies that for each $n \in N_0$, K_n is a subclass of S^*.

We denote by R_n, $n \in N_0$, the subclass of A whose members are characterized by the condition

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \frac{n}{n+1}, \qquad z \in E. \tag{5}$$

It follows immediately that $R_0 \equiv S^*$ and that for each $n \geqslant 1$, $R_n \subset K_n$. Thus, for each $n \in N_0$, R_n is a subclass of S^*. One can readily prove that $R_{n+1} \subset R_n$ for every $n \in N_0$. These inclusion relations and the fact that $z/(1 - xz)$ belongs to R_n if and only if $|x| \leqslant 1/n$, along with a result of Ruscheweyh [3, Corollary 2, Theorem 4] imply that $\bigcap_{n \in N_0} R_n = \{z\}$.

The following interesting results are due to Libera [2].

THEOREM A. *If $f \in S^*$, then so does the function F, defined by*

$$F(z) = \frac{2}{z}\int_0^z f(t)\, dt. \tag{6}$$

THEOREM B. *If $f \in K$, then so does the function F, defined by* (6).

THEOREM C. *If $f \in C$, then so does the function F, defined by* (6).

In this paper, along with other things, we prove that the above-mentioned results of Libera continue to hold for much wider classes than the ones for which he has proved them.

THEOREM 1. *Let $f \in A$ and for a given $n \in N_0$ satisfy the condition*

$$\operatorname{Re}\frac{D^{n+1}f(z)}{D^n f(z)} > \frac{2n-1}{2(n+1)}, \qquad z \in E. \tag{7}$$

Let F be defined by (6). *Then $F \in R_n$.*

PROOF. The condition (7) when expressed in terms of F reads as follows:

$$\operatorname{Re}\left[\frac{(n+2)D^{n+2}F(z)/D^{n+1}F(z) - n}{(n+1) - (n-1)D^n F(z)/D^{n+1}F(z)}\right] > \frac{2n-1}{2(n+1)} \qquad (z \in E), \tag{8}$$

where we have made use of the identity

$$z(D^m F(z))' \equiv (m+1)D^{m+1}F(z) - mD^m F(z) \quad \text{for every } m \in N_0. \tag{9}$$

We have to prove that (8) implies the inequality

$$\operatorname{Re}\frac{D^{n+1}F(z)}{D^nF(z)} > \frac{n}{n+1} \qquad (z \in E).$$

Define w in E by

$$G(z) = \frac{D^{n+1}F(z)}{D^nF(z)} = \frac{n}{n+1} + \frac{1}{n+1}\frac{1-w(z)}{1+w(z)}$$
$$= \frac{(n+1)+(n-1)w(z)}{(n+1)(1+w(z))}. \tag{10}$$

Evidently $w(0) = 0$. Differentiating (10) logarithmically and simplifying, we obtain

$$\left[\frac{(n+2)D^{n+2}F(z)/D^{n+1}F(z) - n}{(n+1)-(n-1)D^nF(z)/D^{n+1}F(z)}\right]$$
$$= \left[\frac{(n+1)+(n-1)w(z)}{(n+1)(1+w(z))} - \frac{1}{n+1}\cdot\frac{zw'(z)}{w(z)}\cdot\frac{w(z)}{1+w(z)}\right]. \tag{11}$$

If $\operatorname{Re} G(z_0) = n/(n+1)$ for a certain z_0 belonging to E and $\operatorname{Re} G(z) > n/(n+1)$ for $|z| < |z_0|$, then $|w(z)| < |w(z_0)| = 1$ for $|z| < |z_0|$ and of course $w(z_0) \neq -1$.

Applying Jack's lemma [1] to $w(z)$ at the point z_0 and letting $z_0w'(z_0)/w(z_0) = k$, so that $k \geq 1$, we obtain from (11)

$$\operatorname{Re}\left[\frac{(n+2)D^{n+2}F(z_0)/D^{n+1}F(z_0) - n}{(n+1)-(n-1)D^nF(z_0)/D^{n+1}F(z_0)}\right]$$
$$= \frac{n}{n+1} - \frac{k}{2(n+1)} \leq \frac{2n-1}{2(n+1)},$$

which contradicts (8). This proves that $\operatorname{Re} G(z) > n/(n+1)$ in E and hence $F \in R_n$. This completes the proof of Theorem 1.

Putting $n = 0$ and $n = 1$ in Theorem 1, we obtain the following interesting results which assert that Theorems A and B of Libera hold under much weaker assumptions.

COROLLARY A. *If f is starlike of order $-1/2$, that is, if $f \in S^*(-1/2)$, then the function F, defined by* (6), *belongs to S^*.*

COROLLARY B. *If f is convex of order $-1/2$, that is, if $f \in K(-1/2)$, then the function F, defined by* (6), *belongs to K.*

It is well known that each $f \in S$ is convex in the disc $|z| < 2 - \sqrt{3}$ and hence, from Theorem B, it follows that the Libera transforms $(2/z)\int_0^z f(t)dt$ of members of S map the disc $|z| < r_0$, $r_0 \geq 2 - \sqrt{3}$, onto convex domains. However, in view of Corollary B, we can strengthen this result.

COROLLARY B′. *If $f \in S$, then the function F defined by* (6) *maps* $|z| < r_1$, $r_1 > 4 - \sqrt{13} = 0.394\ldots$, *onto a convex domain.*

PROOF. It is well known that for $f \in S$, we have

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leqslant \frac{4r}{1-r^2} \qquad (|z| = r < 1),$$

from which we deduce that

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}$$

in $|z| < r_1$, where $r_1 = 4 - \sqrt{13}$, is the smallest positive root of the equation $r^2 - 8r + 3 = 0$. Corollary B′ now follows from Corollary B.

The results of Corollaries A and B were orally communicated to the authors by Professors St. Ruscheweyh and V. Singh.

Our next result shows that Theorem C of Libera holds under much weaker hypotheses.

COROLLARY C. *Let $f \in A$ satisfy the condition*

$$\operatorname{Re}\{f'(z)/g'(z)\} > 0 \quad (z \in E) \tag{12}$$

where g is any member of $K(-1/2)$. Then the function F, defined by (6), *is in C.*

PROOF. Since $g \in K(-1/2)$, from Corollary B it follows that

$$G(z) = \frac{2}{z}\int_0^z g(t)\,dt \in K.$$

From the hypothesis, we have

$$\operatorname{Re}\left\{\frac{zF''(z) + 2F'(z)}{zG''(z) + 2G'(z)}\right\} = \operatorname{Re}\frac{f'(z)}{g'(z)} > 0 \qquad (z \in E). \tag{13}$$

We shall prove that $F(z)$ is close-to-convex with respect to the convex function $G(z)$, that is,

$$\operatorname{Re}\frac{F'(z)}{G'(z)} > 0 \qquad (z \in E). \tag{14}$$

To prove (14), put

$$\frac{F'(z)}{G'(z)} = \frac{1 - w(z)}{1 + w(z)}. \tag{15}$$

The function $w(z)$ defined in this way is clearly regular in E, $w(0) = 0$ and $w(z) \neq -1$ in E. The desired result would follow if we prove that $|w(z)| < 1$ in E.

From (15), we at once obtain

$$\frac{zF''(z) + 2F'(z)}{zG''(z) + 2G'(z)} = \frac{1 - w(z)}{1 + w(z)} - \frac{2zw'(z)}{(1 + w(z))^2}\left(\frac{zG''(z)}{G'(z)} + 2\right)^{-1}. \tag{16}$$

Let us assume that $|w(z)| \nless 1$ in E. Then, by Jack's lemma, there exists a point z_0 in E such that $z_0 w'(z_0) = kw(z_0)$, with $|w(z_0)| = 1$ and $k \geq 1$. Putting $z = z_0$ and $w(z_0) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, in (16), we obtain

$$\frac{z_0 F''(z_0) + 2F'(z_0)}{z_0 G''(z_0) + 2G'(z_0)} = \frac{1 - e^{i\theta}}{1 + e^{i\theta}} - \frac{2ke^{i\theta}}{(1 + e^{i\theta})^2}\left[\frac{z_0 G''(z_0)}{G'(z_0)} + 2\right]^{-1}.$$

Since $\mathrm{Re}(1 - e^{i\theta})/(1 + e^{i\theta}) = 0$, $ke^{i\theta}/(1 + e^{i\theta})^2$ is real and positive and $\mathrm{Re}((zG''(z)/G'(z)) + 1) > 0$ in E, from the last relation a contradiction to our hypothesis (13) would follow. We must therefore have $|w(z)| < 1$ in E and Corollary C is established.

THEOREM 2. *If $f \in R_n$, then the function F, defined by*

$$F(z) = \frac{n+1}{z^n}\int_0^z t^{n-1} f(t)\, dt,$$

belongs to R_{n+1}.

PROOF. From the definition of F, we have

$$\begin{aligned}(n+1)D^n f(z) &= D^n(nF(z) + zF'(z)) \\ &= nD^n F(z) + z(D^n F(z))' \\ &= (n+1)D^{n+1}F(z) \quad \text{(using (9))}.\end{aligned}$$

Similarly $(n+1)D^{n+1}f(z) = (n+2)D^{n+2}F(z) - D^{n+1}F(z)$. From these relations and the fact that $f \in R_n$, we conclude that

$$\mathrm{Re}\left\{\frac{(n+2)D^{n+2}F(z) - D^{n+1}F(z)}{(n+1)D^{n+1}F(z)}\right\} = \mathrm{Re}\frac{D^{n+1}f(z)}{D^n f(z)} > \frac{n}{n+1} \quad (z \in E)$$

from which it follows that

$$\mathrm{Re}\frac{D^{n+2}F(z)}{D^{n+1}F(z)} > \frac{n+1}{n+2} \qquad (z \in E),$$

and therefore $F \in R_{n+1}$.

REMARK. One can show that Theorem 2 remains true on replacing R_n by K_n and R_{n+1} by K_{n+1} in the hypothesis and the conclusion, respectively.

The authors are grateful to the referee for his suggestions which greatly helped in presenting this paper in a compact form.

REFERENCES

1. I. S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc. (2) **3** (1971), 469–474.

2. R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. **16** (1965), 755–758.

3. St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115.

DEPARTMENT OF MATHEMATICS, PUNJABI UNIVERSITY, PATIALA, 147002, PUNJAB, INDIA