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A NOTE ON INVARIANT SUBSPACES FOR FINITE MAXIMAL SUBDIAGONAL ALGEBRAS

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ABSTRACT. Let M be a von Neumann algebra with a faithful, normal, tracial state τ and H^∞ be a finite, maximal, subdiagonal algebra of M. Every left- (or right-) invariant subspace with respect to H^∞ in the noncommutative Lebesgue space $L^p(M, \tau)$, $1 \leqslant p < \infty$, is the closure of the space of bounded elements it contains.

1. Introduction. Let M be a von Neumann algebra with a faithful, normal, tracial state τ and let H^∞ be a finite, maximal, subdiagonal algebra in M. Such algebras were defined and first studied by Arveson [**1**] as noncommutative analogues of weak-*Dirichlet algebras. Since the introduction of these algebras, a number of authors have investigated the structure of the invariant subspaces for H^∞ acting on the noncommutative Lebesgue space $L^p(M, \tau)$ (see, in particular, [**3**], [**5**], [**6**], [**7**] and [**8**]). In [**6**], we showed that, if $\mathfrak{M}$ is a left- (or right-) invariant subspace of $L^p(M, \tau)$, $1 \leqslant p < \infty$, then $\mathfrak{M} \cap M$ contains elements different from zero. In this note, we shall show that, if $\mathfrak{M}$ is a left- (or right-) invariant subspace of $L^p(M, \tau)$, $1 \leqslant p < \infty$, then $\mathfrak{M}$ is the L^p-norm closure of $\mathfrak{M} \cap M$. The method is based on a factorization theorem, i.e. if k is in M with (possibly unbounded) inverse lying in $L^2(M, \tau)$, then there are unitary operators u_1, u_2 in M and operators a_1, a_2 in H^∞ with inverses lying in H^2 such that $k = u_1 a_1 = a_2 u_2$.

2. Let M be a von Neumann algebra with a faithful, normal, tracial state τ. We shall denote the noncommutative Lebesgue spaces associated with M and τ by $L^p(M, \tau)$, $1 \leqslant p < \infty$ ([**2**], [**9**]). As is customary, M will be identified with $L^\infty(M, \tau)$. The closure of a subset S of $L^p(M, \tau)$ in the L^p-norm $\|x\|_p = \tau(|x|^p)^{1/p}$ will be denoted by $[S]_p$.

DEFINITION. Let H^∞ be a σ-weakly closed subalgebra of M containing the identity operator 1 and let Φ be a faithful, normal expectation from M onto $D = H^\infty \cap H^{\infty *}$ ($H^{\infty *} = \{x^*\colon x \in H^\infty\}$). Then H^∞ is called a finite, maximal, subdiagonal algebra in M with respect to Φ and τ in case the following conditions are satisfied: (1) $H^\infty + H^{\infty *}$ is σ-weakly dense in M; (2) $\Phi(xy) = \Phi(x)\Phi(y)$, for all $x, y \in H^\infty$; (3) H^∞ is maximal among those subalgebras of M satisfying (1) and (2); and (4) $\tau \circ \Phi = \tau$.

For $1 \leqslant p < \infty$, the closure of H^∞ in $L^p(M, \tau)$ is denoted by H^p and the

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closure of $H_0^\infty = \{x \in H^\infty; \Phi(x) = 0\}$ is denoted by H_0^p.

In [6], McAsey, Muhly and the author proved that if k is in M with inverse lying in $L^2(M, \tau)$, then there are unitary operators u_1, u_2 in M and operators a_1, a_2 in H^∞ such that $k = u_1a_1 = a_2u_2$. We shall show that in fact it is possible to choose a_1 and a_2 to have inverses lying in H^2.

PROPOSITION 1 (CF. [6, PROPOSITION 1.2]). *If $k \in M$ and $k^{-1} \in L^2(M, \tau)$, then there are unitary operators u_1, $u_2 \in M$ and operators a_1, $a_2 \in H^\infty$ such that $k = u_1a_1 = a_2u_2$ and a_1^{-1}, $a_2^{-1} \in H^2$.*

To Proposition 1, we need the following lemma.

LEMMA 1. *Suppose that $k \in M$ and $k^{-1} \in L^2(M, \tau)$. Then*

(i) $k \notin [kH_0^\infty]_2$.

(ii) *Let η be the projection of k on $[kH_0^\infty]_2$ and $\zeta = k - \eta$. Then there exists a unitary operator u in M such that $u\zeta \in [D]_2$, $[u\zeta D]_2 = [D]_2$ and $uk \in H^\infty$.*

PROOF. See proof of [6, Proposition 1.2].

PROOF OF PROPOSITION 1. Keep the notations in Lemma 1. Put $a = uk$. To prove Proposition 1, it is sufficient to prove that $a^{-1} \in H^2$. If P is the orthogonal projection of $L^2(M, \tau)$ onto $[D]_2$, then the restriction of P to M equals Φ. Since $\eta \in [kH_0^\infty]_2$, there exists a sequence $\{b_n\}_{n=1}^\infty$ in H_0^∞ such that $\lim\|\eta - kb_n\|_2 = 0$. Then we have $u\zeta = u(k - \eta) = \lim(uk - ukb_n) = \lim(a - ab_n)$. Since $u\zeta \in [D]_2$ and $ab_n \in H_0^\infty$, $u\zeta = Pu\zeta = \lim P(a - ab_n) = \lim \Phi(a - ab_n) = \Phi(a)$. It is immediate from this that $u\zeta \in D$. Since $[u\zeta D]_2 = [D]_2$ by Lemma 1 and $a^{-1} = k^{-1}u^*$, we have for every $d \in D$,

$$\begin{aligned}\tau(\Phi(a)P(a^{-1})u\zeta d) &= \tau(P(a^{-1})u\zeta d\Phi(a)) = \tau(a^{-1}u\zeta d\Phi(a))\\ &= \tau(k^{-1}\zeta d\Phi(a)) = \lim \tau\big(k^{-1}(k - kb_n)d\Phi(a)\big)\\ &= \lim \tau((1 - b_n)d\Phi(a)) = \tau(d\Phi(a)) = \tau(u\zeta d).\end{aligned}$$

Consequently we have $\Phi(a)P(a^{-1}) = 1$ and so $a^{-1} = k^{-1}\zeta P(a^{-1})$. For every $d \in D$ and every $x \in H_0^\infty$, we have $\tau(k^{-1}\zeta dx) = \lim \tau(k^{-1}(k - kb_n)dx) = \lim \tau((1 - b_n)\, dx) = 0$. Since $P(a^{-1}) \in [D]_2$, there exists a sequence $\{d_n\}_{n=1}^\infty$ in D such that $\lim\|d_n - P(a^{-1})\|_2 = 0$. Hence, for every $x \in H_0^\infty$, $\tau(a^{-1}x) = \tau(k^{-1}\zeta P(a^{-1})x) = \lim \tau(k^{-1}\zeta d_n x) = 0$. Since $L^2(M, \tau) = H^2 \oplus H_0^{2*}$ ($H_0^{2*} = \{x^*: x \in H_0^2\}$) by [1, p. 583] or [6, Proposition 1.1], we have $a^{-1} \in H^2$. This completes the proof. Q.E.D.

3. In this section, we collect several important facts about H^p and H_0^p.

LEMMA 2. *$H^1 \cap L^2(M, \tau) = H^2$ and $H_0^1 \cap L^2(M, \tau) = H_0^2$.*

PROOF. Since $L^2(M, \tau) = H^2 \oplus H_0^{2*} = H_0^2 \oplus H^{2*}$, this lemma is trivial.

LEMMA 3. *$H^1 = \{x \in L^1(M, \tau): \tau(xy) = 0$ for all $y \in H_0^\infty\}$.*

PROOF. That H^1 is included in the set indicated above is clear. Conversely let $x \in L^1(M, \tau)$ satisfy $\tau(xy) = 0$, $y \in H_0^\infty$. Let $x = |x^*|v$ be the polar decomposition of x. Let f be the function on $[0, \infty)$ defined by the formula $f(t) = 1$, $0 \leq t \leq 1$, $f(t) = 1/t$, $t > 1$, and define k to be $f(|x^*|^{1/2})$ through the functional calculus. Then note that $k \in M$ and $k^{-1} \in L^2(M, \tau)$. By Proposition 1, we may choose a unitary operator u in M and an operator $a \in H^\infty$ such that $k = ua$ and $a^{-1} \in H^2$. Then ax is a nonzero element in $L^2(M, \tau)$. Since $L^2(M, \tau) = H^2 \oplus H_0^{2*}$, we have $ax \in H^2$ and so $x = a^{-1}ax \in H^2H^2 \subset H^1$. This completes the proof.

Since $\|\Phi(x)\|_1 \leq \|x\|_1$ for any x in M, Φ extends uniquely to a projection of norm one of $L^1(M, \tau)$ onto $[D]_1$ and we denote this extension of Φ to $L^1(M, \tau)$ by Φ too. Then we have the following lemma.

LEMMA 4.

$$H_0^1 = \{x \in L^1(M, \tau): \tau(xy) = 0 \textit{ for all } y \in H^\infty\}$$
$$= \{x \in H^1: \Phi(x) = 0\}.$$

PROOF. The inclusion $H_0^1 \subseteq \{x \in L^1(M, \tau): \tau(xy) = 0 \text{ for all } y \in H^\infty\}$ is clear. Now we consider any $x \in L^1(M, \tau)$ such that $\tau(xy) = 0$, $y \in H^\infty$. Since $D \subset H^\infty$, we have $\tau(xy) = \tau(\Phi(x)y) = 0, y \in D$, and so $\Phi(x) = 0$. By Lemma 3, $x \in H^1$. Next we suppose $x \in H^1$ satisfies the equation $\Phi(x) = 0$. Then there exist $x_n \in H^\infty$ such that $\|x_n - x\|_1 \to 0$. Note that $\|x_n - \Phi(x_n) - x\|_1 \to 0$ and $x_n - \Phi(x_n) \in H_0^\infty$. It follows that $x \in H_0^1$. This completes the proof.

PROPOSITION 2. *Let* $1 \leq p \leq \infty$.

(1) $H^1 \cap L^p(M, \tau) = H^p$ *and* $H_0^1 \cap L^p(M, \tau) = H_0^p$.

(2) $H^p = \{x \in L^p(M, \tau): \tau(xy) = 0 \textit{ for all } y \in H_0^\infty\}$.

(3) $H_0^p = \{x \in L^p(M, \tau): \tau(xy) = 0 \textit{ for all } y \in H^\infty\}$.

PROOF. We knew already that this lemma is true for $p = 2$ and for $p = \infty$ (cf. [**1**, Corollary 2.2.4]).

(1) We shall prove the lemma for $1 < p < 2$ using Proposition 1 and for $p > 2$ by a duality argument.

Let $1 < p < 2$. Define the number r by $1/r + 1/2 = 1/p$. It is evident that $H^p \subseteq H^1 \cap L^p(M, \tau)$. To show the reverse inclusion, consider any $x \in H^1 \cap L^p(M, \tau)$. Let $x = |x^*|v$ be the polar decomposition of x. Put $k = f(|x^*|^{p/2})$, where f is the function in the proof of Lemma 3. Then there is an element $a \in H^\infty$ with inverse lying in H^2 such that ax $(\neq 0) \in H^1 \cap L^r(M, \tau)$. Since $L^r(M, \tau) \subset L^2(M, \tau)$, we have

$$ax \in H^1 \cap L^r(M, \tau) \subset H^1 \cap L^2(M, \tau) = H^2 \subset H^p.$$

So

$$x = a^{-1}ax \in H^2ax \subset [H^\infty ax]_p \subset H^p.$$

It follows that $H^p = H^1 \cap L^p(M, \tau)$ in this case. $H_0^p = H_0^1 \cap L^p(M, \tau)$ in the case $1 < p < 2$ may be proved in just the same way.

Let $2 < p < \infty$. Here again the inclusion $H^p \subset H^1 \cap L^p(M, \tau)$ is trivial. It is sufficient to show that if $y \in L^q(M, \tau)$ where $1/p + 1/q = 1$ and $y \perp H^p$, i.e. $\tau(yx) = 0$, $x \in H^p$, then $y \perp H^1 \cap L^p(M, \tau)$. Now the relation $y \perp H^p$ implies by Lemma 4 that $y \in H_0^1 \cap L^q(M, \tau) = H_0^q$, as $1 < q < 2$. So there exist $y_n \in H_0^\infty$ such that $\|y_n - y\|_q \to 0$. This means that $0 = \tau(y_n x) \to \tau(yx)$ for all $x \in H^1 \cap L^p(M, \tau)$. So $y \perp H^1 \cap L^p(M, \tau)$.

(2) and (3) are clear by (1) and Lemmas 3 and 4. This completes the proof.

4. Let $\mathfrak{M}$ be a closed subspace of $L^p(M, \tau)$. We shall say that $\mathfrak{M}$ is left- (resp. right-) invariant if $H^\infty \mathfrak{M} \subseteq \mathfrak{M}$ (resp. $\mathfrak{M} H^\infty \subseteq \mathfrak{M}$). Our goal in this note is the following theorem.

THEOREM. *Let $\mathfrak{M}$ be a left- (or right-) invariant subspace of $L^p(M, \tau)$, $1 \leqslant p < \infty$. Then $\mathfrak{M}$ is the closure of the space of bounded operators it contains.*

PROOF. (1) Case $2 \leqslant p < \infty$. Define the number q by the equation $1/p + 1/q = 1$. If $[\mathfrak{M} \cap M]_p \subsetneq \mathfrak{M}$, then there exist an element $\xi \in \mathfrak{M}$ and $x \in L^q(M, \tau)$ such that $\tau(\xi x) \neq 0$ and $\tau(yx) = 0$ for every $y \in [\mathfrak{M} \cap M]_p$. Let $\xi = |\xi^*| v$ be the polar decomposition of ξ. Since $\xi \in L^p(M, \tau) \subset L^2(M, \tau)$, we may form $k = f(|\xi^*|)$, where f is the function in the proof of Lemma 3. Note that $k \in M$ and $k^{-1} \in L^p(M, \tau) \subset L^2(M, \tau)$. By Proposition 1, we may choose a unitary operator u in M and an operator $a \in H^\infty$ such that $k = ua$ and $a^{-1} \in H^2$. By Proposition 2, $a^{-1} \in L^p(M, \tau) \cap H^2 = H^p$ and note that $a\xi$ is a nonzero element in $\mathfrak{M} \cap M$. Since $\mathfrak{M}$ is left-invariant, we have $ba\xi \in \mathfrak{M} \cap M$ for every $b \in H^\infty$ and so $\tau(ba\xi x) = 0$. By Proposition 2, $a\xi x \in H_0^p$. Therefore $\tau(\xi x) = \tau(a^{-1} a \xi x) = 0$. This is a contradiction.

(2) Case $1 \leqslant p < 2$. Define the number q and r by the equations $1/p + 1/q = 1$ and $1/r + 1/2 = 1/p$. If $[\mathfrak{M} \cap M]_p \subsetneq \mathfrak{M}$, then there exist $\xi \in \mathfrak{M}$ and $x \in L^q(M, \tau)$ such that $\tau(\xi x) \neq 0$ and $\tau(yx) = 0$ for every $y \in [\mathfrak{M} \cap M]_p$. Let $\xi = |\xi^*| v$ be the polar decomposition of ξ. Put $k = f(|\xi^*|^{p/2})$, where f is the function in the proof of Lemma 3. By Proposition 1, there is an element $a \in H^\infty$ with inverse lying in H^2 such that $a\xi$ $(\neq 0) \in L^r(M, \tau) \cap \mathfrak{M} \subset L^2(M, \tau) \cap \mathfrak{M}$. As in (1), there exists an element $b \in H^\infty$ with inverse lying in H^r such that $ba\xi$ $(\neq 0) \in \mathfrak{M} \cap M$. For every $c \in H^\infty$, we have $cba\xi \in \mathfrak{M} \cap M$ and so $\tau(cba\xi x) = 0$. By Proposition 2, $ba\xi x \in H_0^q$. Since $(ba)^{-1} = a^{-1} b^{-1} \in H^2 H^r \subset H^p$, we have $\tau(\xi x) = \tau((ba)^{-1} ba \xi x) = 0$. This is a contradiction.

This completes the proof.

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