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ULTRA-STRONG DITKIN SETS IN HYPERGROUPS

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ABSTRACT. We introduce compactly separating sets for hypergroups which under certain conditions turn out to be ultra-strong Ditkin for the hypergroup algebra whenever their boundary is. We also characterize such sets for the hypergroup related to p-adic numbers.

1. Introduction. Wik [13] defined and studied strong Ditkin sets in the circle group T as those spectral sets E in T for which there exists an approximate identity in the kernel $k(E)$ of E in $l^1(Z)$. Rosenthal [9] carried this study further to some locally compact abelian groups Γ, in particular, he proved that every closed coset in Γ is strong Ditkin and a nowhere dense strong Ditkin set is a member of the discrete coset ring $R(\Gamma_d)$. Gilbert [5] proved that closed sets in $R(\Gamma_d)$ are Calderon and Schreiber [10] showed that every such set is strong Ditkin thus completing the characterization of strong Ditkin sets with empty interior. Liu, Rooij and Wang [8] showed that a closed ideal I in $L^1(G)$ has a bounded approximate identity if and only if I is the kernel of a closed element E of $R(\Gamma_d)$ where Γ is the dual group of G. Rosenthal [9] proved that if both G and Γ are metrizable then E is strong Ditkin whenever the boundary Bd E of E is. However·his method does not force E to be ultra-strong Ditkin even if Bd E is. On the contrary, it follows from the above discussion that every proper closed interval I in the additive group $\mathbf{R}$ is strong Ditkin because its boundary is ultra-strong Ditkin whereas I cannot be ultra-strong Ditkin. A study of such notions for hypergroup algebras was begun in [1], [2]. Let K be a locally compact commutative hypergroup with Haar measure m [3], [7], [11], [12] whose dual $\hat{K}$ is a hypergroup and equals the set $\chi_b(K)$ of bounded continuous characters on K, $L^1(m) = L^1(K)$ the convolution algebra and $A(\hat{K})$ the algebra of Fourier transforms $\hat{f}$ of f in $L^1(K)$. As in [1], a closed subset E of $\hat{K}$ will be called *strong Ditkin* if there exists a net $\{f_\alpha: \alpha \in D\}$ in $L^1(K)$ such that

(i) for each α, $\hat{f}_\alpha = 0$ in a neighborhood of E and has compact support,

(ii) $\sup\{\|f_\alpha\|_E: \alpha \in D\} < \infty$, where

$$\|f_\alpha\|_E = \sup\{\|f_\alpha * f\|_1: f \in J(E), \|f\|_1 \leqslant 1\}$$

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and

(iii) for $f \in k(E)$, $f * f_\alpha \to f$ in $L^1(K)$, where $J(E)$ is the closure of $\{f \in L^1(K)\colon \hat{f}$ vanishes in a neighborhood of $E\}$ and $k(E) = \{f \in L^1(K)\colon \hat{f}$ vanishes on $E\}$.

E will be called *ultra-strong Ditkin* if it satisfies (i) and (iii) of above and $\sup\{\|f_\alpha\|_1\colon \alpha \in D\} < \infty$.

The empty set and the members of the center $Z(\hat{K})$ of $\hat{K}$ are all ultra-strong Ditkin, other points of $\hat{K}$ need not even be spectral. Nothing is known about the status of subhypergroups of $\hat{K}$. In this paper we introduce a class of closed subsets E of $\hat{K}$, called compactly separating, for which E is ultra-strong Ditkin whenever Bd E is. This class includes the classes of open subhypergroups of $\hat{K}$ whose complements are compact and of complements of compact open subhypergroups of $\hat{K}$. This immediately gives that all such hypergroups are ultra-strong Ditkin. We characterize this class (and also the ultra-strong Ditkin sets) for the example given by Dunkl and Ramirez **[4]** and further prove that not every closed subset is ultra-strong Ditkin even though every closed subset is strong Ditkin **[4]**, **[2]**, **[1]**. We shall freely use the notations, terminology and results on spectral synthesis for the hypergroup algebra $L^1(K)$ from **[1]**, **[2]**. In particular, K will be a commutative hypergroup whose dual $\hat{K}$ is a hypergroup and equals $\chi_b(K)$. We just remark that most of the results can be reformulated even when $\hat{K}$ is not $\chi_b(K)$ à la Theorem 3.3 **[2]**. Further their analogues for Segal algebras based on **[1]** can also be proved.

2. Definition 1. Let E and F be closed subsets of $\hat{K}$ and $\alpha > 1$. F will be said to be *α-boundedly disjoint* from E if there exists a symmetric neighborhood V of 1 with compact closure such that

(i) $(F * V * V) \cap E = \Phi$ and

(ii) $\pi(F * V)/\pi(V) \leq \alpha$.

DEFINITION 2. A closed subset E of $\hat{K}$ will be called *compactly separating* if for some $\alpha > 1$, every compact subset of $\hat{K} \setminus E$ is α-boundedly disjoint from E.

LEMMA 3. *Let F be a compact subset of $\hat{K}$ which is α^2-boundedly disjoint from a closed subset E of $\hat{K}$. Then there exists a $\varphi \in A_{00}(\hat{K})$ such that φ is* 1 *on F, zero on E, $0 \leq \varphi \leq 1$ and $\|\varphi\|_A \leq \alpha$.*

PROOF. Choose V as in Definition 1 and then apply Lemma 2.5 **[2]**.

THEOREM 4. *Let E be a compactly separating subset of $\hat{K}$. If* Bd E *is ultra-strong Ditkin then so is E.*

PROOF. Because of Lemma 3, Rosenthal's proof of the corresponding result viz. Theorem 2.4(b) **[9]** for strong Ditkin sets can be modified to give this result.

REMARK 5. We just note that ultra-strong Ditkin sets need not be compactly separating. For instance it can be easily seen that no point of the group T is compactly separating.

COROLLARY 6. *Let* E_0 *denote the set of points of* $\hat{K}$ *which are ultra-strong Ditkin. If* $\hat{K}$ *is discrete at points of* $\hat{K} \setminus F_0$ *for some finite subset* F_0 *of* E_0 *then every compactly separating subset of* $\hat{K}$ *is ultra-strong Ditkin.*

PROOF. It follows immediately from the above theorem since finite unions of ultra-strong Ditkin sets are ultra-strong Ditkin [**1**, Remark 3.4(vi)].

REMARK 7. The corresponding results for Calderon (F_0 can be even countable) and strong Ditkin sets can be easily formulated.

THEOREM 8. *Let* H *be an open subhypergroup of* $\hat{K}$.

(i) *if* H *is compact then* $\hat{K} \setminus H$ *is compactly separating,*

(ii) *if* $\hat{K} \setminus H$ *is compact then* H *is compactly separating.*

PROOF. (i) Take $V = H$. For any compact subset F of H, $F * V * V = F * V \subset H$ and thus F is 1-boundedly disjoint from $\hat{K} \setminus H$.

(ii) Let V be any compact neighborhood of 1 contained in H. By [7, 10.3A] $(\hat{K} \setminus H) * H * H = \hat{K} \setminus H$.

Let $\alpha = \pi(\hat{K} \setminus H)/\pi(V)$. So for any compact subset F of $\hat{K} \setminus H$, $(F * V * V) \cap H = \Phi$ and $\pi(F * V)/\pi(V) \leqslant \alpha$. Thus F is α-boundedly disjoint from H.

THEOREM 9. (i) *The complement of a compact open subhypergroup* H *in* $\hat{K}$ *is ultra-strong Ditkin.*

(ii) *An open subhypergroup with a compact complement is ultra-strong Ditkin.*

PROOF. The sets in both the cases are closed and have empty boundary. Further Φ is ultra-strong Ditkin by Remark 3.4(ii) [**1**], which in fact, follows immediately from Theorem 2.8 of [**2**] on the existence of a bounded approximate identity in $L^1(K)$.

EXAMPLE 10. Let p be a prime number and $a = 1/p$. Then the hypergroup H_a is defined by Dunkl and Ramirez [**4**] to be the one point compactification Z_+^* of the set Z_+ of positive integers with Haar measure m given by

$$\begin{aligned} m(k) &= (1-a)a^k, \quad k \neq \infty, \\ &= 0, \quad k = \infty, \end{aligned}$$

and convolution given by

$$p_n * p_m = p_{\min(n,m)} \quad \text{for } n, m \in Z_+^* \text{ and } n \neq m, \text{ for } n \in Z_+,$$

$$p_n * p_n(t) = \begin{cases} 0, & t < n, \\ \dfrac{1-2a}{1-a}, & t = n, \\ a^k, & t = n + k > n, \end{cases}$$

and $p_\infty * p_\infty = p_\infty$.

As explained in Example 4.6 [2] H_a can be considered as $\hat{K}$, where K is $\hat{H}_a$ and members of $\hat{H}_a$ are given by $\{\chi_n: n \in Z_+\}$ where

$$\chi_n(k) = \begin{cases} 0, & k < n-1, \\ a/(a-1), & k = n-1, \\ 1, & k \geqslant n \text{ or } k = \infty. \end{cases}$$

(i) A closed subset E of $\hat{K}$ is compactly separating if and only if either E is finite and $\infty \notin E$ or $\hat{K} \setminus E$ is finite if and only if E is open.

(ii)(a) All closed open subsets of $\hat{K}$ are ultra-strong Ditkin.

(b) All finite subsets of $\hat{K}$ are ultra-strong Ditkin.

(c) An infinite closed subset of $\hat{K}$ whose complement is infinite is not ultra-strong Ditkin.

Equivalently,

(ii)′ A closed subset E of $\hat{K}$ is ultra-strong Ditkin if and only if it is finite or open.

PROOF. (i)(a) Let $E \subset \hat{K}$ be closed such that $\hat{K} \setminus E$ is finite. Then $\infty \in E$.

Let $k_0 = \max\{k: k \in \hat{K} \setminus E\}$, $V = \{k \in \mathbf{N}: k > k_0\} \cup \{\infty\}$ and $\alpha = 1/m(V)$. Then for any compact subset F of $\hat{K} \setminus E$, $F * V * V = F * V = F$, also $m(F) \leqslant 1$ and therefore, F is α-boundedly disjoint from E. Hence E is compactly separating.

(b) Let $E \subset \hat{K}$ be finite and $\infty \notin E$. Let $k_0 = \max\{k: k \in E\}$, $V = \{k \in \mathbf{N}: k > k_0\} \cup \{\infty\}$ and $\alpha = 1/m(V)$. Let $B = \{k \in Z_+: k \leqslant k_0\} \setminus E$. Then $\hat{K} \setminus E = V \cup B$. So

$$(\hat{K} \setminus E) * V * V = (V \cup B) * V * V = V \cup B = \hat{K} \setminus E.$$

So for any compact subset F of $\hat{K} \setminus E$, $(F * V * V) \cap E = \Phi$ and $m(F * V)/m(V) \leqslant \alpha$. Hence E is compactly separating.

(c) Let E be a closed subset of $\hat{K}$ such that it is not of any of the above two types.

Then $\hat{K} \setminus E$ is infinite and $\infty \in E$. Let $\hat{K} \setminus E = \{k_i: i \in \mathbf{N}\}$, where $\{k_i\}$ is a strictly increasing sequence of positive integers. Let, if possible, E be compactly separating and α a bound. Then $F_j = \{k_i: 1 \leqslant i \leqslant j\}$ is a compact subset of $\hat{K} \setminus E$ and therefore, there exists a (symmetric) neighborhood V_j of 1 such that $(F_j * V_j * V_j) \cap E = \Phi$ and $m(F_j * V_j)/m(V_j) \leqslant \alpha$. If $V_j \not\subset \{k: k > k_j\}$ then there is a $p \leqslant k_j$ such that $p \in V_j$. There are two possibilities

(i) $p \in F_j \cap V_j$: Then

$$\{k \in \mathbf{N}: k > p\} \subset p * p \subset F_j * V_j \subset F_j * V_j * V_j.$$

But E is infinite so $\{k \in \mathbf{N}: k > p\} \cap E \neq \Phi$, which contradicts $(F_j * V_j * V_j) \cap E = \Phi$.

(ii) $p \in V_j \setminus F_j$, then $p \in E$. Also $p \in k_j * p \subset F_j * V_j$. So $p \in F_j * V_j * V_j$, which is a contradiction. Hence $V_j \subset \{k: k > k_j\}$. So $m(V_j)$ is less than or equal to a^{k_j+1}. Also $F_j * V_j = F_j$. So

$$m(F_j * V_j) = m(F_j) = \sum_{i=1}^{j} a^{k_i}(1-a) \geqslant j a^{k_j}(1-a).$$

So $m(F_j * V_j)/m(V_j) > ja^{k_j}(1-a)/a^{k_j+1} = j(1-a)/a$.

So $j < \alpha a/(1-a)$, a contradiction. Hence E is not compactly separating subset of $\hat{K}$.

(d) The last equivalence now follows immediately because a closed set E in $\hat{K}$ is open if and only if E is finite and $\infty \notin E$ or $\hat{K} \setminus E$ is finite and $\infty \in E$.

(ii)(a) Follows from (i) and Theorem 5 above.

(b) If E is finite and $\infty \notin E$ then E is ultra-strong Ditkin by (i). If E is finite and $\infty \in E$ then $E \setminus \{\infty\}$ is ultra-strong Ditkin by the above argument. Also $\{\infty\}$ is ultra-strong Ditkin by Theorem 3.3 **[2]** and therefore, by Remark 3.4(vi) **[1]**, E is ultra-strong Ditkin.

(c) Let, if possible, E be ultra-strong Ditkin, since E is infinite and $\hat{K} \setminus E$ is infinite there exists a sequence $\{\alpha_n\}$ in N such that $\alpha_n < \alpha_n + 1 < \alpha_{n+1}$, $\alpha_n \in \hat{K} \setminus E$ and $\alpha_n + 1 \in E$ for all n. Define φ on $\hat{K}$ by $\varphi(\alpha_n) = 1/n^2$ and zero otherwise. Then $\varphi \in A(\hat{K})$ by Theorem 7.8 **[4]**. Further $\varphi = 0$ on E. Let $\varphi = \hat{f}$ for $f \in L^1(K)$ then $f \in k(E)$. Since E is ultra-strong Ditkin, $k(E)$ has factorization by Cohen's factorization theorem. So there exists $g,h \in k(E)$ such that $f = g * h$.

Then $\hat{g}(\alpha_n + 1) = 0$ for each n. So by Theorem 7.8 **[4]**

$$\sum |\hat{g}(\alpha_n)| \leq \sum |\hat{g}(n) - \hat{g}(n-1)| < \infty.$$

Similarly $\sum |\hat{h}(\alpha_n)| < \infty$. Also $1/n^2 = \hat{f}(\alpha_n) = \hat{g}(\alpha_n)\hat{h}(\alpha_n)$.

So by the Cauchy-Schwarz inequality

$$\begin{aligned}\sum \frac{1}{n} &= \sum \sqrt{|\hat{g}(\alpha_n)|}\sqrt{|\hat{h}(\alpha_n)|}\\ &\leq \left(\sum |\hat{g}(\alpha_n)|\right)^{1/2}\left(\sum |\hat{h}(\alpha_n)|\right)^{1/2}\\ &< \infty, \quad \text{a contradiction.}\end{aligned}$$

Hence E is not ultra-strong Ditkin.

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