ULTRA-STRONG DITKIN SETS IN HYPERGROUPS

AJIT KAUR CHILANA AND AJAY KUMAR

Abstract. We introduce compactly separating sets for hypergroups which under certain conditions turn out to be ultra-strong Ditkin for the hypergroup algebra whenever their boundary is. We also characterize such sets for the hypergroup related to \( p \)-adic numbers.

1. Introduction. Wik [13] defined and studied strong Ditkin sets in the circle group \( T \) as those spectral sets \( E \) in \( T \) for which there exists an approximate identity in the kernel \( k(E) \) of \( E \) in \( L^1(Z) \). Rosenthal [9] carried this study further to some locally compact abelian groups \( \Gamma \), in particular, he proved that every closed coset in \( \Gamma \) is strong Ditkin and a nowhere dense strong Ditkin set is a member of the discrete coset ring \( R(\Gamma_d) \). Gilbert [5] proved that closed sets in \( R(\Gamma_d) \) are Calderón and Schreiber [10] showed that every such set is strong Ditkin thus completing the characterization of strong Ditkin sets with empty interior. Liu, Rooij and Wang [8] showed that a closed ideal \( I \) in \( L^1(G) \) has a bounded approximate identity if and only if \( I \) is the kernel of a closed element \( E \) of \( R(\Gamma_d) \) where \( \Gamma \) is the dual group of \( G \). Rosenthal [9] proved that if both \( G \) and \( \Gamma \) are metrizable then \( E \) is strong Ditkin whenever the boundary \( Bd \ E \) of \( E \) is. However, his method does not force \( E \) to be ultra-strong Ditkin even if \( Bd \ E \) is. On the contrary, it follows from the above discussion that every proper closed interval \( I \) in the additive group \( \mathbb{R} \) is strong Ditkin because its boundary is ultra-strong Ditkin whereas \( I \) cannot be ultra-strong Ditkin. A study of such notions for hypergroup algebras was begun in [1], [2]. Let \( K \) be a locally compact commutative hypergroup with Haar measure \( m \) [3], [7], [11], [12] whose dual \( \hat{K} \) is a hypergroup and equals the set \( \chi_0(K) \) of bounded continuous characters on \( K \), \( L^1(m) = L^1(K) \) the convolution algebra and \( A(\hat{K}) \) the algebra of Fourier transforms \( \hat{f} \) of \( f \) in \( L^1(K) \). As in [1], a closed subset \( E \) of \( \hat{K} \) will be called strong Ditkin if there exists a net \( \{f_\alpha; \alpha \in D\} \) in \( L^1(K) \) such that

(i) for each \( \alpha, \hat{f}_\alpha = 0 \) in a neighborhood of \( E \) and has compact support,

(ii) \( \sup\{\|f_\alpha\|_E; \alpha \in D\} < \infty \), where

\[
\|f_\alpha\|_E = \sup\{\|f_\alpha \ast f\|_1; f \in J(E), \|f\|_1 \leq 1\}
\]

Received by the editors August 30, 1978 and, in revised form, December 14, 1978.


Key words and phrases. Hypergroup, ultra-strong Ditkin set, strong Ditkin set, compactly separating set.

© 1979 American Mathematical Society

0002-9939/79/0000-0561/502.50

353
and

(iii) for $f \in k(E), f \cdot f_a \to f$ in $L^1(K)$, where $J(E)$ is the closure of \{ $f \in L^1(K): \hat{f}$ vanishes in a neighborhood of $E$ \} and $k(E) = \{ f \in L^1(K): \hat{f}$ vanishes on $E$ \}.

$E$ will be called ultra-strong Ditkin if it satisfies (i) and (iii) of above and $\text{sup}\{||f_a||_1: a \in D\} < \infty$.

The empty set and the members of the center $Z(\hat{K})$ of $\hat{K}$ are all ultra-strong Ditkin, other points of $\hat{K}$ need not even be spectral. Nothing is known about the status of subhypergroups of $\hat{K}$. In this paper we introduce a class of closed subsets $E$ of $\hat{K}$, called compactly separating, for which $E$ is ultra-strong Ditkin whenever $\text{Bd} E$ is. This class includes the classes of open subhypergroups of $\hat{K}$ whose complements are compact and of complements of compact open subhypergroups of $\hat{K}$. This immediately gives that all such hypergroups are ultra-strong Ditkin. We characterize this class (and also the ultra-strong Ditkin sets) for the example given by Dunkl and Ramirez [4] and further prove that not every closed subset is ultra-strong Ditkin even though every closed subset is strong Ditkin [4], [2], [1]. We shall freely use the notations, terminology and results on spectral synthesis for the hypergroup algebra $L^1(K)$ from [1], [2]. In particular, $K$ will be a commutative hypergroup whose dual $\hat{K}$ is a hypergroup and equals $\chi_0(K)$. We just remark that most of the results can be reformulated even when $K$ is not $\chi_0(K)$ à la Theorem 3.3 [2]. Further their analogues for Segal algebras based on [1] can also be proved.

2. Definition 1. Let $E$ and $F$ be closed subsets of $\hat{K}$ and $\alpha > 1$. $F$ will be said to be $\alpha$-boundedly disjoint from $E$ if there exists a symmetric neighborhood $V$ of $1$ with compact closure such that

(i) $(F \ast V \ast V) \cap E = \emptyset$ and

(ii) $\pi(F \ast V)/\pi(V) < \alpha$.

Definition 2. A closed subset $E$ of $\hat{K}$ will be called compactly separating if for some $\alpha > 1$, every compact subset of $\hat{K} \setminus E$ is $\alpha$-boundedly disjoint from $E$.

Lemma 3. Let $F$ be a compact subset of $\hat{K}$ which is $\alpha^2$-boundedly disjoint from a closed subset $E$ of $\hat{K}$. Then there exists a $\varphi \in A_{\omega}(\hat{K})$ such that $\varphi$ is 1 on $F$, zero on $E$, $0 < \varphi < 1$ and $||\varphi||_A < \alpha$.

Proof. Choose $V$ as in Definition 1 and then apply Lemma 2.5 [2].

Theorem 4. Let $E$ be a compactly separating subset of $\hat{K}$. If $\text{Bd} E$ is ultra-strong Ditkin then so is $E$.

Proof. Because of Lemma 3, Rosenthal's proof of the corresponding result viz. Theorem 2.4(b)[9] for strong Ditkin sets can be modified to give this result.
Remark 5. We just note that ultra-strong Ditkin sets need not be compactly separating. For instance it can be easily seen that no point of the group $T$ is compactly separating.

Corollary 6. Let $E_0$ denote the set of points of $\hat{K}$ which are ultra-strong Ditkin. If $\hat{K}$ is discrete at points of $\hat{K} \setminus F_0$ for some finite subset $F_0$ of $E_0$ then every compactly separating subset of $\hat{K}$ is ultra-strong Ditkin.

Proof. It follows immediately from the above theorem since finite unions of ultra-strong Ditkin sets are ultra-strong Ditkin [1, Remark 3.4(vi)].

Remark 7. The corresponding results for Calderón ($F_0$ can be even countable) and strong Ditkin sets can be easily formulated.

Theorem 8. Let $H$ be an open subhypergroup of $\hat{K}$.
(i) if $H$ is compact then $\hat{K} \setminus H$ is compactly separating,
(ii) if $\hat{K} \setminus H$ is compact then $H$ is compactly separating.

Proof. (i) Take $V = H$. For any compact subset $F$ of $H$, $F \star V \star V = F \star V \subset H$ and thus $F$ is 1-boundedly disjoint from $\hat{K} \setminus H$.

(ii) Let $V$ be any compact neighborhood of 1 contained in $H$. By [7, 10.3A] $(\hat{K} \setminus H) \star H \star H = \hat{K} \setminus H$.

Let $\alpha = \pi(\hat{K} \setminus H)/\pi(V)$. So for any compact subset $F$ of $\hat{K} \setminus H$, $(F \star V \star V) \cap H = \Phi$ and $\pi(F \star V)/\pi(V) < \alpha$. Thus $F$ is $\alpha$-boundedly disjoint from $H$.

Theorem 9. (i) The complement of a compact open subhypergroup $H$ in $\hat{K}$ is ultra-strong Ditkin.
(ii) An open subhypergroup with a compact complement is ultra-strong Ditkin.

Proof. The sets in both the cases are closed and have empty boundary. Further $\Phi$ is ultra-strong Ditkin by Remark 3.4(ii) [1], which in fact, follows immediately from Theorem 2.8 of [2] on the existence of a bounded approximate identity in $L^1(K)$.

Example 10. Let $p$ be a prime number and $a = 1/p$. Then the hypergroup $H_a$ is defined by Dunkl and Ramirez [4] to be the one point compactification $Z^*_+ \subset Z^*_+$ of the set $Z^*_+$ of positive integers with Haar measure $m$ given by

$$m(k) = (1 - a)a^k, \quad k \neq \infty,$$
$$= 0, \quad k = \infty,$$

and convolution given by

$$p_n \star p_m = p_{\min(n,m)} - \hat{E} \text{ for } n, m \in Z^*_+, \text{ and } n \neq m, \text{ for } n \in Z^*_+,\text{ and }$$

$$p_n \star p_n(t) = \begin{cases} 0, & t < n, \\ 1 - 2a, & t = n, \\ \frac{1 - a}{1 - a}, & t = n + k > n, \\ a^k, & t = n + k > n, \end{cases}$$

and $p_\infty \star p_\infty = p_\infty$. 
As explained in Example 4.6 [2] $H_a$ can be considered as $\hat{K}$, where $K$ is $\hat{H}_a$ and members of $\hat{H}_a$ are given by $\{x_n: n \in Z_+\}$ where

$$x_n(k) = \begin{cases} 0, & k < n - 1, \\ a/(a - 1), & k = n - 1, \\ 1, & k > n \text{ or } k = \infty. \end{cases}$$

(i) A closed subset $E$ of $\hat{K}$ is compactly separating if and only if either $E$ is finite and $\infty \notin E$ or $\hat{K} \setminus E$ is finite if and only if $E$ is open.

(ii)(a) All closed open subsets of $\hat{K}$ are ultra-strong Ditkin.

(b) All finite subsets of $\hat{K}$ are ultra-strong Ditkin.

(c) An infinite closed subset of $\hat{K}$ whose complement is infinite is not ultra-strong Ditkin.

Equivalently,

(ii)' A closed subset $E$ of $\hat{K}$ is ultra-strong Ditkin if and only if it is finite or open.

Proof. (i)(a) Let $E \subset \hat{K}$ be closed such that $\hat{K} \setminus E$ is finite. Then $\infty \in E$.

Let $k_0 = \max\{k: k \in \hat{K} \setminus E\}$, $V = \{k \in N: k > k_0\} \cup \{\infty\}$ and $\alpha = 1/m(V)$. Then for any compact subset $F$ of $\hat{K} \setminus E$, $F * V * V = F * V = F$, also $m(F) < 1$ and therefore, $F$ is $\alpha$-boundedly disjoint from $E$. Hence $E$ is compactly separating.

(b) Let $E \subset \hat{K}$ be finite and $\infty \notin E$. Let $k_0 = \max\{k: k \in E\}$, $V = \{k \in N: k > k_0\} \cup \{\infty\}$ and $\alpha = 1/m(V)$. Let $B = \{k \in Z_+: k < k_0\} \setminus E$. Then $\hat{K} \setminus E = V \cup B$. So

$$\hat{K} \setminus E = V \cup B.$$

So for any compact subset $F$ of $\hat{K} \setminus E$, $(F * V * V) \cap E = \emptyset$ and $m(F * V)/m(V) < \alpha$. Hence $E$ is compactly separating.

(c) Let $E$ be a closed subset of $\hat{K}$ such that it is not of any of the above two types.

Then $\hat{K} \setminus E$ is infinite and $\infty \in E$. Let $\hat{K} \setminus E = \{k_i: i \in N\}$, where $\{k_i\}$ is a strictly increasing sequence of positive integers. Let, if possible, $E$ be compactly separating and $\alpha$ a bound. Then $F_j = \{k_i: 1 < i < j\}$ is a compact subset of $\hat{K} \setminus E$ and therefore, there exists a (symmetric) neighborhood $V_j$ of 1 such that $(F_j * V_j * V_j) \cap E = \emptyset$ and $m(F_j * V_j)/m(V_j) < \alpha$. If $V_j \subset \{k: k > k_j\}$ then there is a $p < k_j$ such that $p \in V_j$. There are two possibilities

(i) $p \in F_j \cap V_j$; Then

$$\{k \in N: k > p\} \subset p * p \subset F_j * V_j \subset F_j * V_j * V_j.$$

But $E$ is infinite so $\{k \in N: k > p\} \cap E = \emptyset$, which contradicts $(F_j * V_j * V_j) \cap E = \emptyset$.

(ii) $p \in V_j \setminus F_j$, then $p \in E$. Also $p \in k_j * p \subset F_j * V_j$. So $p \in F_j * V_j * V_j$, which is a contradiction. Hence $V_j \subset \{k: k > k_j\}$. So $m(V_j)$ is less than or equal to $a^{k_j+1}$. Also $F_j * V_j = F_j$. So

$$m(F_j * V_j) = m(F_j) = \sum_{i=1}^{j} a_i(1 - a) > ja^{k_j+1}(1 - a).$$
So \( m(F_j \ast V_j)/m(V_j) > j\alpha/(1 - \alpha) \), a contradiction. Hence \( E \) is not compactly separating subset of \( \hat{K} \).

(d) The last equivalence now follows immediately because a closed set \( E \) in \( \hat{K} \) is open if and only if \( E \) is finite and \( \infty \notin E \) or \( \hat{K} \setminus E \) is finite and \( \infty \in E \).

(ii)(a) Follows from (i) and Theorem 5 above.

(b) If \( E \) is finite and \( \infty \notin E \) then \( E \) is ultra-strong Ditkin by (i). If \( E \) is finite and \( \infty \in E \) then \( E \setminus \{\infty\} \) is ultra-strong Ditkin by the above argument. Also \( \{\infty\} \) is ultra-strong Ditkin by Theorem 3.3 [2] and therefore, by Remark 3.4(vi) [1], \( E \) is ultra-strong Ditkin.

(c) Let, if possible, \( E \) be ultra-strong Ditkin, since \( E \) is infinite and \( \hat{K} \setminus E \) is infinite there exists a sequence \( \{\alpha_n\} \) in \( \mathbb{N} \) such that \( \alpha_n < \alpha_{n+1} \), \( \alpha_n \in \hat{K} \setminus E \) and \( \alpha_n + 1 \in E \) for all \( n \). Define \( \varphi \) on \( \hat{K} \) by \( \varphi(\alpha_n) = 1/n^2 \) and zero otherwise. Then \( \varphi \in A(\hat{K}) \) by Theorem 7.8 [4]. Further \( \varphi = 0 \) on \( E \). Let \( \varphi = \hat{f} \) for \( f \in L^1(K) \) then \( f \in k(E) \). Since \( E \) is ultra-strong Ditkin, \( k(E) \) has factorization by Cohen's factorization theorem. So there exists \( g, h \in k(E) \) such that \( f = g \ast h \).

Then \( \hat{g}(\alpha_n + 1) = 0 \) for each \( n \). So by Theorem 7.8 [4]

\[
\sum |\hat{g}(\alpha_n)| < \sum |\hat{f}(n) - \hat{g}(n - 1)| < \infty.
\]

Similarly \( \sum |\hat{h}(\alpha_n)| < \infty \). Also \( 1/n^2 = \hat{f}(\alpha_n) = \hat{g}(\alpha_n)\hat{h}(\alpha_n) \).

So by the Cauchy-Schwarz inequality

\[
\sum \frac{1}{n} = \sum \sqrt{|\hat{g}(\alpha_n)|} \sqrt{|\hat{h}(\alpha_n)|}
< \left( \sum |\hat{g}(\alpha_n)| \right)^{1/2} \left( \sum |\hat{h}(\alpha_n)| \right)^{1/2}
< \infty, \quad \text{a contradiction.}
\]

Hence \( E \) is not ultra-strong Ditkin.

We thank the referee for his useful comments and suggestions.

REFERENCES


Department of Mathematics, University of Delhi, Delhi-110007, India