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COMMON FIXED POINTS AND PARTIAL ORDERS

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ABSTRACT. It is observed that certain theorems on common fixed points may be derived from a theorem on partially ordered sets.

In a previous note [2] we explained how certain fixed point theorems are direct consequences of certain theorems on partially ordered sets. In fact, let $(E, \leqslant)$ be a partially ordered set which admits at least one maximal element x_0, and let f be a self-mapping of E such that $x \leqslant f(x)$ for all $x \in E$; then $f(x_0) = x_0$. In the present note (which is entirely conceptual) we shall develop this point of view a little further; our motivation has mainly been the appearance of [6].

NOTATION. Everywhere in the following (E, d) is a *complete* metric space, $\leqslant$ is a partial order on E (i.e., $\leqslant$ is reflexive, transitive and asymmetric), and φ is a real valued function on E which is *bounded below*. For $x \in E$ we denote by $S(x, \leqslant)$ the set of points $y \in E$ such that $x \leqslant y$.

We shall base our considerations on the following variant of Theorem 1 of [1]:

PROPOSITION. *Assume that*

(a) *φ is decreasing with respect to $\leqslant$, i.e., $x \leqslant y$ implies $\varphi(x) \geqslant \varphi(y)$;*

(b) *for all $\epsilon > 0$ there exists $\delta > 0$ such that $x \leqslant y$ and $\varphi(x) - \varphi(y) < \delta$ implies $d(x, y) < \epsilon$.*

Then there exists a sequence $(x_n)_{n \in \mathbf{N}}$ in E (where x_1 may be taken arbitrary) and a point $x_0 \in E$ such that

(c) *$x_n \leqslant x_{n+1}$ for all $n \in \mathbf{N}$, and $x_n \to x_0$;*

(d) *$y_n \to x_0$ for all sequences $(y_n)_{n \in \mathbf{N}}$ with $x_n \leqslant y_n$.*

Furthermore,

(e) *if $x_n \leqslant x_0$ for all $n \in \mathbf{N}$, then x_0 is maximal in $(E, \leqslant)$.*

PROOF. We define $(x_n)_{n \in \mathbf{N}}$ inductively. Take $x_1 \in E$ arbitrary. When $x_1, x_2, \ldots, x_n$ have been chosen, let $a_n := \inf \varphi(S(x_n, \leqslant))$, and take $x_{n+1} \in S(x_n, \leqslant)$ such that $\varphi(x_{n+1}) \leqslant a_n + n^{-1}$. Then $x_n \leqslant x_{n+1}$, and for any $y \in S(x_n, \leqslant)$ we have

$$a_{n-1} \leqslant a_n \leqslant \varphi(y) \leqslant \varphi(x_n) \leqslant a_{n-1} + (n-1)^{-1}. \qquad (*)$$

In particular, $0 \leqslant \varphi(x_n) - \varphi(x_m) \leqslant (n-1)^{-1}$ for $n \leqslant m$. Using (b) we then

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see that $(x_n)_{n \in \mathbf{N}}$ is a Cauchy sequence, which by completeness converges, $x_n \to x_0$. Hence, (c) is proved. Using (b) it also follows from (*) that the diameter of $S(x_n, <)$ tends to 0 as n tends to ∞. Therefore, we also have $y_n \to x_0$ when $y_n \in S(x_n, <)$, whence (d) is proved. Finally, suppose that $x_n < x_0$ for all $n \in \mathbf{N}$, and let $y \in E$ be such that $x_0 < y$. Then we also have $x_n < y$ for all $n \in \mathbf{N}$ by (c). Taking $y_n := y$ for all $n \in \mathbf{N}$, (d) then shows that $y = x_0$, i.e., x_0 is maximal. So (e) is proved.

REMARK 1. The proof of the Proposition is based on a standard argument, cf. Remark 1 of **[1]** and the proof of Lemma 1.1 of **[5]**. The argument also appears in **[6]** and **[7]**.

NOTATION. Define a partial order $\leqslant_{d,\varphi}$ on E by letting $x \leqslant_{d,\varphi} y$ if and only if $d(x, y) \leqslant \varphi(x) - \varphi(y)$, cf. **[1]**, **[2]**. Call a self-mapping f of E *admissible* if $x \leqslant_{d,\varphi} f(x)$ for all $x \in E$, and call a family F of self-mappings of E *admissible* if each $f \in F$ is admissible. For a set F of self-mappings of E define a relation $\leqslant_F$ on E by letting $x \leqslant_F y$ if and only if $x = y$ or $y = f_n \circ \cdots \circ f_1(x)$ for suitable $f_1, \ldots, f_n \in F$. (Clearly, $\leqslant_F$ is reflexive and transitive. If F is admissible, then $\leqslant_F$ is finer than $\leqslant_{d,\varphi}$, i.e., $x \leqslant_F y$ implies $x \leqslant_{d,\varphi} y$, and therefore in this case $\leqslant_F$ is also asymmetric, and hence a partial order.) For a set F of self-mappings of E let F^* denote the set of finite compositions $f_n \circ \cdots \circ f_1$ of mappings $f_1, \ldots, f_n \in F$. (Note that $x \leqslant_F y$ if and only if $x \leqslant_{F^*} y$. Also note that if F is closed under finite compositions, i.e., $F = F^*$, then $x \leqslant_F y$ if and only if $x = y$ or $y = f(x)$ for some $f \in F$.) Call F^* *closed under countable compositions* if for each sequence $(f_n^*)_{n \in \mathbf{N}}$ of mappings $f_n^* \in F^*$ and each $x \in E$ such that $f_n^*(x) \to y$ for some $y \in E$, there exists $g^* \in F^*$ such that $g^*(x) = y$.

THEOREM. *Let F be an admissible set of self-mappings of E. Assume that at least one of the following conditions holds*:

(1) *φ is lower semicontinuous.*

(2) *F^* is closed under countable compositions.*

(3) *Each $f \in F$ is continuous.*

Then there is a common fixed point x_0 for all mappings $f \in F$.

PROOF. In case (1) we shall apply the Proposition to the partial order $\leqslant_{d,\varphi}$. We first note that (a) and (b) hold for $\leqslant_{d,\varphi}$ by the definition of $\leqslant_{d,\varphi}$. Therefore, the Proposition is applicable, let $(x_n)_{n \in \mathbf{N}}$ and x_0 be as described in (c)–(e). Now, note that the sets $S(x, \leqslant_{d,\varphi})$ are closed by (1). Therefore, since $x_m \in S(x_n, \leqslant_{d,\varphi})$ for $m \geqslant n$, it follows that $x_0 \in S(x_n, \leqslant_{d,\varphi})$, i.e., $x_n \leqslant_{d,\varphi} x_0$ for all $n \in \mathbf{N}$. By (e) it next follows that x_0 is maximal in $(E, \leqslant_{d,\varphi})$. Then x_0 is also maximal in $(E, \leqslant_F)$, since $\leqslant_F$ is finer than $\leqslant_{d,\varphi}$. This shows that $f(x_0) = x_0$ for all $f \in F$.

In cases (2) and (3) we shall apply the Proposition to the partial order $\leqslant_F$. Since (a) and (b) hold for $\leqslant_{d,\varphi}$, and $\leqslant_F$ is finer than $\leqslant_{d,\varphi}$, (a) and (b) also hold for $\leqslant_F$. So, the Proposition is applicable, let $(x_n)_{n \in \mathbf{N}}$ and x_0 be as described in (c)–(e).

In case (2), suppose that there exists a subsequence $(x'_n)_{n\in\mathbf{N}}$ of $(x_n)_{n\in\mathbf{N}}$ such that $x'_n \neq x'_{n+1}$ for all $n \in \mathbf{N}$. Then for each n there exists $f^*_n \in F^*$ such that $x'_{n+1} = f^*_n(x'_n)$, whence

$$x'_{n+p} = f^*_{n+p-1} \circ \cdots \circ f^*_n(x'_n)$$

for all $n, p \in \mathbf{N}$. Since $x'_{n+p} \to x_0$ for $p \to \infty$, it follows from (2) that there exists $g^*_n \in F^*$ such that $g^*_n(x'_n) = x_0$, whence $x'_n \prec_F x_0$. But then clearly we also have $x_n \prec_F x_0$ for all $n \in \mathbf{N}$. On the other hand, if no such subsequence $(x'_n)_{n\in\mathbf{N}}$ exists, then $x_{m+p} = x_m$ for some $m \in \mathbf{N}$ and all $p \in \mathbf{N}$. Since $x_{m+p} \to x_0$ for $p \to \infty$, we see that $x_{m+p} = x_0$ for some m and all p. But then also in this case we have $x_n \prec_F x_0$ for all $n \in \mathbf{N}$. By (e) we then see that x_0 is maximal in $(E, \prec_F)$, and therefore $f(x_0) = x_0$ for all $f \in F$.

In case (3), let $f \in F$, and take $y_n := f(x_n)$. By (d) we then have $f(x_n) \to x_0$. But we also have $f(x_n) \to f(x_0)$ by (c) and the assumption (3). Therefore, $f(x_0) = x_0$ for all $f \in F$.

REMARK 2. Case (1) is only a slight extension of the so-called Caristi's fixed point theorem, see [2], [3], [4]. Note that x_0 is a common fixed point for *all* admissible mappings.

REMARK 3. In case (2), for any given point $x_1 \in E$ there is a common fixed point x_0 such that $x_0 = f_n \circ \cdots \circ f_1(x_1)$ for suitable $f_1, \ldots, f_n \in F$. In fact, it follows from the Proposition that one can obtain $x_1 \prec_F x_0$. In particular, if $F = F^*$, then there is $f \in F$ such that $x_0 = f(x_1)$. This yields Theorem 1.6(a) of [6].

REMARK 4. In case (3), for any given point $x_1 \in E$ there is a common fixed point x_0 such that x_0 is the limit of a sequence $(f^*_n(x_1))_{n\in\mathbf{N}}$, where $f^*_n \in F^*$; this follows from the Proposition. If in addition $F = F^*$, then there is a sequence $(f_n)_{n\in\mathbf{N}}$ with $f_n \in F$ such that $f_n(x_1) \to x_0$. This yields Theorem 1.6 (b) of [6].

REMARK 5. The existence of a fixed point x_0 for a *simple continuous* admissible mapping f follows easily by a direct argument. In fact, since $f^n(x) \prec_{d,\varphi} f^m(x)$ when $n \leqslant m$, and φ is decreasing with respect to $\prec_{d,\varphi}$, we see that the sequence $(\varphi(f^n(x)))_{n\in\mathbf{N}}$ is a decreasing sequence in $\mathbf{R}$. Since φ is bounded below, we have $\varphi(f^n(x)) \to a$, where $a := \inf_{n\in\mathbf{N}} \varphi(f^n(x))$. Therefore, since $d(f^n(x), f^m(x)) \leqslant \varphi(f^n(x)) - \varphi(f^m(x))$ for $n \leqslant m$, it follows that $(f^n(x))_{n\in\mathbf{N}}$ is a Cauchy sequence. By completeness then there is $x_0 \in E$ such that $f^n(x) \to x_0$. By continuity of f we then also have $f^n(x) \to f(x_0)$, i.e., $f(x_0) = x_0$. This should be compared with §1.7 of [6].

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