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ON CLOSED IMAGES OF ORTHOCOMPACT SPACES

GARY GRUENHAGE

ABSTRACT. An example is given which shows that orthocompactness is not preserved by closed maps. We also investigate a property, weaker than orthocompactness, which is preserved by closed maps.

1. Introduction. A collection $\mathfrak{U}$ of open subsets of a topological space X is called a *Q-collection* if $\bigcap \mathfrak{U}'$ is open whenever $\mathfrak{U}' \subset \mathfrak{U}$. If a Q-collection $\mathfrak{U}$ refines some open cover $\mathfrak{V}$ of X, $\mathfrak{U}$ is called a *Q-refinement* of $\mathfrak{V}$. A space X is *orthocompact* if every open cover has a Q-refinement. For many interesting results on orthocompactness, as well as a curious relationship with normality, see B. Scott [S_1].

Scott asks whether orthocompactness is preserved by perfect maps, or any "reasonable" kind of map. It is natural to consider the question for closed maps, since many covering properties are preserved by closed maps. In fact, H. Junnila [J] has shown that if the closed image of an orthocompact space is θ-refinable, then it is also orthocompact.

In this paper we give an example which shows that, in general, orthocompactness is not preserved by closed maps. The question of whether orthocompactness is preserved by perfect maps remains open.[1] We also investigate a property, weaker than orthocompactness, which is preserved by closed maps. This property seems to be useful in eliminating some possible counterexamples to the "perfect map question", and we also use it to improve a theorem of Scott.

2. The example. We shall give an example of an orthocompact space X and a closed map f: $X \to Y$, where Y is not orthocompact. Both X and Y are collectionwise normal. The space Y is obtained from the following "machine" for constructing nonorthocompact spaces from nonmetacompact ones. This machine is a generalization of a machine of R. W. Heath and W. F. Lindgren [HL] which produces nonorthocompact spaces from noncountably metacompact ones.

LEMMA 2.1. *Let X be any nonmetacompact space. Then there exists a space X^* such that X^* is not orthocompact, and if X is normal, collectionwise normal, screenable, or θ-refinable, so is X^*.*

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[1]Dennis Burke has recently answered this question in the negative.

PROOF. Let $\mathcal{U}$ be an open cover of X with no point-finite refinement. Let $X^* = X \cup (\bigcup\{X_u: u \in \mathcal{U}\})$, where X_u is a copy of X, the points of X_u are isolated, and the points in X have basic neighborhoods of the form $V^*_{\mathcal{F}} = V \cup \{V_u: u \in \mathcal{U} \setminus \mathcal{F}\}$, where $\mathcal{F}$ is some finite subset of $\mathcal{U}$, and V_u is a copy of V in X_u. Note that X^* is simply $X \times D^*_k$, where D^*_k is the one-point compactification of the discrete space of cardinality $k = \text{card}(\mathcal{U})$, but with points of the form (x, y) isolated if y is isolated in D^*_k. In fact, $X \times D^*_k$ is not orthocompact either (same proof as below), but this product does not preserve all the nice properties of X that X^* does.

We will show that the open cover $\mathcal{U}^* = \{U^*_{\{U\}}: U \in \mathcal{U}\} \cup \{U_U: U \in \mathcal{U}\}$ of X^* has no Q-refinement. Suppose, on the contrary, that $\mathcal{W}$ is a strict Q-refinement of $\mathcal{U}^*$. Let $W_U \in \mathcal{W}$ be such that $W_U \subset U^*_{\{U\}}$. The collection $\{W_U \cap X: U \in \mathcal{U}\}$ covers X and refines $\mathcal{U}$, so there exists a point $x \in X$ and $U_1, U_2, \ldots$ such that $x \in \bigcap_{i=1}^{\infty} W_{U_i}$. Now if $O^*_{\mathcal{F}}$ is a basic open set containing x, then there exists i_0 such that $U_{i_0} \notin \mathcal{F}$. Thus the copy of the point x in $X_{U_{i_0}}$ is in $O^*_{\mathcal{F}}$, but not in $W_{U_{i_0}}$. Thus $\bigcap_{i=1}^{\infty} W_{U_i}$ is not open, so X^* is not orthocompact. It is easy to see that X^* preserves the properties claimed.

Now for the example. Let $Y = \omega_1 \times \omega_1$ with the topology generated by

(i) all points of the form (α, β), $\beta > 0$;

(ii) sets of the form $(\alpha, \beta] \times (\omega_1 \setminus F)$, where F is a finite subset of ω_1.

Note that Y is simply the machine of Lemma 2.1 applied to ω_1. Thus Y is not orthocompact.

Let $X = \{(\alpha, \beta, \gamma) \in \omega_1^3: \gamma < \max\{\alpha, \beta\}\}$ with the topology generated by

(i) all points in X of the form (α, β, γ), $\gamma > 0$;

(ii) sets of the form $[(\alpha, \beta] \times (\alpha', \beta'] \times (\omega_1 \setminus F)] \cap X$, where F is a finite subset of ω_1. Notice that X is a subspace of the machine of Lemma 2.1 applied to $\omega_1 \times \omega_1$, with only countably many isolated points "above" each point of $\omega_1 \times \omega_1$. It turns out that X is orthocompact, and the projection map $f: X \to Y$ defined by $f(\alpha, \beta, \gamma) = (\alpha, \gamma)$ is a closed map of X onto Y.

Proof that X is orthocompact. Let $\mathcal{U}$ be an open cover of X. For each $\alpha \in \omega_1$, let $[(\alpha', \alpha] \times (\alpha'', \alpha] \times (\omega_1 \setminus F_\alpha)] \cap X$ be a basic open set containing $(\alpha, \alpha, 0)$ and contained in some $U \in \mathcal{U}$. The maps $f_i: \omega_1 \to \omega_1$ defined by $f_1(\alpha) = \alpha'$ and $f_2(\alpha) = \alpha''$ are regressive, so there exists an uncountable set A of ω_1, and ordinals α'_0 and α''_0 such that $f_1(\alpha) = \alpha'_0$ and $f_2(\alpha) = \alpha''_0$ for all $\alpha \in A$. Applying the Δ-system lemma to $\{F_\alpha: \alpha \in A\}$, we conclude that there exists an uncountable set $A' \subset A$ and a set G such that $F_\alpha \cap F_\beta = G$ whenever $\alpha, \beta \in A'$, $\alpha \neq \beta$.

Now pick $\alpha_0 \in A'$. We can choose $\alpha_1 \in A'$, $\alpha_1 > \alpha_0$, such that $F_{\alpha_1} \setminus G \subset \omega_1 \setminus (\alpha_0 + 1)$. Suppose $\alpha_\beta \in A'$ has been defined for all $\beta < \gamma < \omega_1$. Let $\delta = \sup\{\alpha_\beta: \beta < \gamma\}$. Then choose $\alpha_\gamma \in A'$, $\alpha_\gamma > \delta$, such that $F_{\alpha_\gamma} \setminus G \subset \omega_1 \setminus (\delta + 1)$.

Now let $V_\gamma = [(\alpha'_\gamma, \alpha_\gamma] \times (\alpha''_\gamma, \alpha_\gamma] \times (\omega_1 \setminus F_{\alpha_\gamma})] \cap X$, and notice that $V_\gamma \subset V_{\gamma'}$ whenever $\gamma < \gamma'$. Thus $\mathcal{V} = \{V_\gamma: \gamma \in \omega_1\}$ is a Q-collection covering "most" of X. We still must cover $[0, \alpha_0'] \times \omega_1 \times \{0\}$ and $\omega_1 \times [0, \alpha_0''] \times \{0\}$ with Q-collections, and throw in a few isolated points, and then we will be done.

For each $\beta \in \omega_1$, there exists a finite cover of $[0, \alpha_0'] \times \{\beta\} \times \{0\}$ of the form

$$\left\{\left[(\alpha_i(\beta), \alpha_i'(\beta)\right] \times (\beta', \beta] \times (\omega_1 \setminus G_\beta)\right] \cap X : i = 1, 2, \ldots, n_\beta\}$$

which refines $\mathcal{U}$, where $\alpha_i(\beta) < \alpha_0'$ and $\alpha_i'(\beta) \leqslant \alpha_0'$. The map $\beta \to \beta'$ is regressive, so there exists $\beta_0' \in \omega_1$ and an uncountable set $T \subset \omega_1$ such that $\beta \to \beta_0'$ whenever $\beta \in T$. Applying the Δ-system lemma, there exists a set H and an uncountable set $T' \subset T$ such that $G_\alpha \cap G_\beta = H$ whenever $\alpha, \beta \in T'$, $\alpha \neq \beta$. As above, we can construct by induction an uncountable set $T'' \subset T'$ such that $G_\beta \setminus H \subset \omega_1 \setminus (\alpha + 1)$ whenever $\alpha, \beta \in T''$, $\alpha < \beta$. And finally, there exists an uncountable set $T''' \subset T''$ such that $n_\alpha = n_\beta = n$, $\alpha_i(\alpha) = \alpha_i(\beta)$, and $\alpha_i'(\alpha) = \alpha_i'(\beta)$ whenever $\alpha, \beta \in T'''$ and $i \leqslant n$.

The collection

$$\mathcal{O}(i) = \left\{\left[(\alpha_i(\beta), \alpha_i'(\beta)\right] \times (\beta_0', \beta] \times (\omega_1 \setminus G_\beta)\right] \cap X : \beta \in T'''\}$$

is well-ordered by inclusion, and hence is a Q-collection. Now $\mathcal{U} \cup \mathcal{O}(1) \cup \cdots \cup \mathcal{O}(n)$ covers all points of $\omega_1^2 \times \{0\}$ except $\omega_1 \times [0, \max\{\beta_0', \alpha_0''\}] \times \{0\}$. Treating this set in the same way as above, we cover all but a compact subset of $\omega_1^2 \times \{0\}$ by a finite number of Q-collections. Add a finite cover of this compact set, throw in the isolated points, and we have a Q-refinement of $\mathcal{U}$. Thus X is orthocompact.

Proof that the map f: $X \to Y$ defined by $f((\alpha, \beta, \gamma)) = (\alpha, \gamma)$ is closed and continuous. Since $f^{-1}((\alpha, \gamma))$ is a collection of isolated points for $\gamma > 0$, and

$$f^{-1}((\alpha', \alpha] \times \omega_1 \setminus F) = \left[(\alpha', \alpha] \times \omega_1 \times (\omega_1 \setminus F)\right] \cap X,$$

we see that f is continuous. Suppose H is a closed subset of X, but $f(H)$ is not closed in Y. Since Y is a Fréchet space, there exists $\{(\alpha_n, \gamma_n)\}_{n=1}^\infty \subset f(H)$ with $(\alpha_n, \gamma_n) \to (\alpha, 0) \notin f(H)$. Thus there exist $(\alpha_n, \beta_n, \gamma_n) \in H$, $n = 1, 2, \ldots$. It is easy to see that if β is a cluster point of $\{\beta_n\}_{n=1}^\infty$, then $(\alpha, \beta, 0)$ is a cluster point of $\{(\alpha_n, \beta_n, \gamma_n)\}_{n=1}^\infty$. Thus $(\alpha, \beta, 0) \in H$, and so $(\alpha, 0) \in f(H)$, a contradiction. This finishes the proof.

The above example can easily be modified to show that orthocompactness is not preserved by quasi-perfect maps (maps which are closed and have countably compact fibers). Let $X' = X \cup \{c_{\alpha,\gamma} : \alpha, \gamma \in \omega_1, \gamma > 0\}$. Define basic open sets in X' as follows. For points other than the $c_{\alpha,\gamma}$, the basic open neighborhoods are the same as in X. A basis for $c_{\alpha,\gamma}$ is $\{\{c_{\alpha,\gamma}\} \cup (f^{-1}(\alpha, \gamma) \setminus F) : F$ is a finite subset of $f^{-1}(\alpha, \gamma)\}$. Define g: $X' \to Y$ by $g|_X = f$, and $g(c_{\alpha,\gamma}) = (\alpha, \gamma)$. It is easily seen that X' is orthocompact and g is quasi-perfect.

3. Pointwise star-orthocompactness. In this section we introduce a property which we call "pointwise star-orthocompactness". This property is preserved by closed images of orthocompact spaces. It seems to be useful in eliminating some spaces as possible closed or perfect images of orthocompact spaces, and we also use it to improve a theorem of Scott [S_3].

DEFINITION. X is *pointwise star-orthocompact* if for every open cover $\mathcal{U}$ of X, there exists an open Q-collection $\mathcal{V} = \{V_x: x \in X\}$ such that $x \in V_x \subset \operatorname{st}(x, \mathcal{U})$.

REMARK. This is stronger than saying there exists a Q-refinement of $\{\operatorname{st}(x, \mathcal{U}): x \in X\}$. For example, $\omega_1 \times (\omega_1 + 1)$ is countably compact, and so there is a finite subcover from $\{\operatorname{st}(x, \mathcal{U}): x \in X\}$. But it turns out, as shown below, that $\omega_1 \times (\omega_1 + 1)$ is not pointwise star-orthocompact.

The following lemma is straightforward, so we omit its proof.

LEMMA 3.1. *Let $f: X \to Y$ be a closed map. For $U \subset X$, define $f^*(U) = \{y \in Y: f^{-1}(y) \subset U\}$. Then if $\mathcal{U}$ is a Q-collection in X, the collection $\{f^*(U): U \in \mathcal{U}\}$ is a Q-collection in Y.*

THEOREM 3.2. *Pointwise star-orthocompactness is preserved by closed maps.*

PROOF. Let $f: X \to Y$ be a closed map, with X pointwise star-orthocompact. Suppose $\mathcal{U}$ is an open cover of Y. Let $\{V_x: x \in X\}$ be a Q-collection in X such that $x \in V_x \subset \operatorname{st}(x, f^{-1}(\mathcal{U}))$. For each $y \in Y$, let $W_y = f^*(\bigcup\{V_x: x \in f^{-1}(y)\})$. Since f is closed, W_y is an open set containing y. If $z \in W_y$, then

$$f^{-1}(z) \subset \bigcup \{V_x: x \in f^{-1}(y)\} \subset \operatorname{st}(f^{-1}(y), f^{-1}(\mathcal{U}))$$
$$= \bigcup \{f^{-1}(U): y \in U \in \mathcal{U}\}.$$

Thus $z \in \operatorname{st}(y, \mathcal{U})$, and so $W_y \subset \operatorname{st}(y, \mathcal{U})$, which proves that Y is pointwise star-orthocompact.

Now we look at situations in which pointwise star-orthocompactness is equivalent to orthocompactness.

THEOREM 3.3. *A pointwise star-orthocompact developable space is orthocompact.*

PROOF. Let X be a pointwise star-orthocompact space with development $\mathcal{U}_1, \mathcal{U}_2, \ldots$. Applying the pointwise star-orthocompactness to the $\mathcal{U}_n$ we see that every open cover of X has a refinement which is the countable union of Q-collections. Since X is perfect, hence countably metacompact, X is therefore orthocompact [S_1].

An immediate corollary is that the perfect image of an orthocompact developable space is orthocompact, since perfect maps preserve developability. However, Junnila's theorem mentioned in the introduction implies the much stronger result that the closed image of an orthocompact, θ-refinable space is orthocompact. Thus, a nonorthocompact, θ-refinable space cannot be the closed image of an orthocompact space. Our next theorem shows that the same is true for a finite product of locally compact LOTS (linearly ordered topological spaces).

THEOREM 3.4. *A finite product of locally compact LOTS is pointwise star-orthocompact if and only if it is orthocompact.*

PROOF. To prove this, it turns out we can follow the proof of Scott's theorem in [S_1] that orthocompactness is equivalent to normality for finite products or ordinals, and then follow his proof in [S_2] that the same is true for finite products of locally compact LOTS. The only part which does not carry over trivially to pointwise star-orthocompactness is the lemma that if κ is a regular cardinal, $\kappa > \omega$, then $\kappa \times (\kappa + 1)$ is not orthocompact. So all we will prove here is that $\kappa \times (\kappa + 1)$ is not pointwise star-orthocompact.

For each $\alpha < \kappa$, let $U_\alpha = [0, \alpha] \times (\alpha, \kappa]$. Let $\mathcal{U} = \{\kappa \times [0, \kappa)\} \cup \{U_\alpha: \alpha < \kappa\}$. Suppose $\kappa \times (\kappa + 1)$ is pointwise star-orthocompact. Then for each $\alpha < \kappa$, there exists an open set V_α such that $(\alpha, \kappa) \in V_\alpha \subset \mathrm{st}((\alpha, \kappa), \mathcal{U})$, and $\{V_\alpha: \alpha < \kappa\}$ is a Q-collection. Notice that $V_\alpha \cap (\kappa \times [0, \alpha]) = \varnothing$. For each $\alpha < \kappa$, there exists $\alpha' < \alpha$ such that $(\alpha', \alpha] \times \{\kappa\} \subset V_\alpha$. Since $\alpha \to \alpha'$ is a regressive function, there exist a set S cofinal in κ and $\alpha_0 < \kappa$ such that $\alpha \to \alpha_0$ whenever $\alpha \in S$. Thus, $(\alpha_0 + 1, \kappa) \in \bigcap_{\alpha \in S} V_\alpha$, but $(\bigcap_{\alpha \in S} V_\alpha) \cap (\kappa \times \kappa) = \varnothing$. This contradicts the fact that $\{V_\alpha: \alpha < \kappa\}$ is a Q-collection. Thus $\kappa \times (\kappa + 1)$ is not pointwise star-orthocompact.

DEFINITION. A subset A of a space X is said to be *Q-embedded* in X if every open cover of A has a Q-refinement which covers A.

THEOREM 3.5. *Closed paracompact subsets of pointwise star-orthocompact spaces are Q-embedded.*

PROOF. Let A be a closed paracompact subspace of a pointwise star-orthocompact space X. Let $\mathcal{U}$ be an open cover of A. Let $\mathcal{V}$ be a locally finite (in A) refinement of $\mathcal{U}$ by relatively open subsets of A. For each $V \in \mathcal{V}$, pick O_V open in X such that $O_V \cap A = V$, and O_V is contained in some element of $\mathcal{U}$. Let $\mathcal{W}$ be a star-refinement (in A) of $\mathcal{V}$ such that each element of $\mathcal{W}$ hits only finitely many elements of $\mathcal{V}$. For each $W \in \mathcal{W}$, let P_W be open in X such that

(i) $P_W \cap A = W$, and

(ii) $P_W \subset \bigcap \{O_V: V \in \mathcal{V} \text{ and } W \subset V\}$.

Let $\mathcal{W}^* = \{P_W: W \in \mathcal{W}\}$. Suppose $x \in A$. There exists $V_0 \in \mathcal{V}$ such that $\mathrm{st}(x, \mathcal{W}) \subset V_0$. So if $x \in W \in \mathcal{W}$, then $W \subset V_0$, so $P_W \subset O_{V_0}$. Thus, $\mathrm{st}(x, \mathcal{W}^*) \subset O_{V_0}$, so the collection $\{\mathrm{st}(x, \mathcal{W}^*): x \in A\}$ refines $\mathcal{U}$.

Let $\mathcal{G} = \{G_x: x \in X\}$ witness the pointwise star-orthocompactness of X for the cover $\mathcal{W}^* \cup \{X \setminus A\}$. Then $\{G_x: x \in A\}$ is a Q-collection refining $\mathcal{U}$ and covering A.

COROLLARY 3.6. *If $X = X_1 \cup X_2$, where X_1 is closed and paracompact, and X_2 is open and orthocompact, then X is orthocompact if and only if X is pointwise star-orthocompact.*

The last corollary shows that we can replace the hypothesis of "ultraparacompact" in a theorem of Scott by "paracompact".

COROLLARY 3.7. *Let $f\colon X \to Y$ be a closed map with X orthocompact, and suppose $D = \{y \in Y\colon |f^{-1}(y)| > 1\}$ has paracompact closure in Y. Then Y is orthocompact.*

PROOF. Suppose $\mathcal{U}$ is an open cover of Y. By Theorem 3.5, there is a refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}_{\bar{D}} = \{V \in \mathcal{V}\colon V \cap \bar{D} \neq \varnothing\}$ is a Q-collection. Let $\mathcal{W}$ be a Q-refinement in X of $\{f^{-1}(V)\colon V \in \mathcal{V}\}$. Then

$$\{V \in \mathcal{V}\colon V \cap \bar{D} \neq \varnothing\} \cup \{f(W)\colon W \in \mathcal{W}, W \cap \bar{D} = \varnothing\}$$

is a Q-refinement of $\mathcal{U}$ covering Y.

CONCLUDING REMARKS. We have seen that the closed image of an orthocompact space must be pointwise star-orthocompact. Junnila [J] has shown that it must also be *discretely orthocompact*, i.e., whenever $\mathcal{F}$ is a discrete collection of closed sets, and, for each $F \in \mathcal{F}$, V_F is an open neighborhood of F, then there is a Q-collection $\{W_F\colon F \in \mathcal{F}\}$ such that $F \subset W_F \subset V_F$ for all $F \in \mathcal{F}$. In fact, Junnila proves that discrete orthocompactness is preserved by closed maps, and that θ-refinable discretely orthocompact spaces are orthocompact, to obtain the result mentioned in the introduction. Scott [S_1] has shown that a perfect image of an orthocompact space must be *weakly orthocompact*, i.e., for every open cover $\mathcal{U}$, there is a Q-refinement of the set of all finite unions of elements of $\mathcal{U}$. We should like to point out that the space Y defined in §2 has all three of the above generalized orthocompactness properties, but is not orthocompact. So it would be interesting to know if there is some reason why Y *cannot* be a perfect image of an orthocompact space.[2] Of course, we have tried to modify our example to make the fibers compact, but without success.

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DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36830

[2]Burke's example shows that Y is the perfect image of an orthocompact space.