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THE GENERATION OF NONLINEAR EQUIVARIANT DIFFERENTIAL OPERATORS[1]

ROBERT DELVER

ABSTRACT. Finite generation results are given for the set of smooth nonlinear differential operators: $C^\infty(M, N) \to C^\infty(M, \mathbf{R})$ of order $\leq k$ which are equivariant with respect to the action of a Lie group on the base manifold M.

1. Introduction. Let G be a Lie group acting by diffeomorphisms ϕ_g, $g \in G$, on a smooth manifold M, N a smooth manifold and let $\mathfrak{D}_k$, $k \in \{\infty, 1, 2, 3, \ldots\}$, denote the real vector space of smooth nonlinear differential operators of order $\leq k$ of $C^\infty(M, N)$ into $C^\infty(M, \mathbf{R})$. The action of G on M lifts to $C^\infty(M, N)$ by $g \cdot f = f \circ \phi_g^{-1}$, $f \in C^\infty(M, N)$, $g \in G$. Let $\mathfrak{D}_k^G$ denote the G-equivariant elements of $\mathfrak{D}_k$. Full definitions are given in §2.

There are two equivariance preserving structures on $\mathfrak{D}_\infty$ each with its own generation problem. The first structure is a multiplication: $\mathfrak{D}_k \times \mathfrak{D}_k \to \mathfrak{D}_k$, defined by

$$\mathfrak{F}_1 \cdot \mathfrak{F}_2(f) = \mathfrak{F}_1(f)\mathfrak{F}_2(f), \ f \in C^\infty(M, N). \tag{1.1}$$

If $N = \mathbf{R}$, a second structure is induced by the composition $\mathfrak{D}_{k_1} \times \mathfrak{D}_{k_2} \to \mathfrak{D}_{k_1+k_2}$, given by

$$\mathfrak{F}_1\mathfrak{F}_2(f) = \mathfrak{F}_1(\mathfrak{F}_2(f)), \ f \in C^\infty(M, \mathbf{R}). \tag{1.2}$$

The main results of this paper are two finiteness theorems, one for each of these structures.

THEOREM 1. *Let G be a compact Lie group, M a smooth G-manifold of finite orbit type and N a smooth manifold then, for each $k \in \{0, 1, 2, \ldots\}$, there exist $\mathfrak{A}_1, \ldots, \mathfrak{A}_i \in \mathfrak{D}_k^G$ such that $\mathfrak{F} \in \mathfrak{D}_k^G$ iff $\mathfrak{F} = f(\mathfrak{A}_1, \ldots, \mathfrak{A}_i)$, for some $f \in C^\infty(\mathbf{R}^i)$.*

This theorem is based on a theorem of Schwarz **[10]**, the proof is given in §2.

Let $C^\infty(M)^G$ denote the G-invariant elements of $C^\infty(M)$. A function ξ: $M \to \mathbf{R}^l$ is called a finite generator for $C^\infty(M)^G$ iff $C^\infty(M)^G = \xi^* C^\infty(M)$.

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We let $\Gamma^G(T(M))$ denote the $C^\infty(M)^G$-module of invariant vector fields on M. In the case where M is a principal G-bundle finite generators exist both for $C^\infty(M)^G$ and for the module $\Gamma^G(T(M))$ (Lemmas (3.1) and (3.2)). Moreover, $\{X_1, \ldots, X_n\} \subset \Gamma^G(T(M))$ is a generator for $\Gamma^G(T(M))$ iff $\{X_1(x), \ldots, X_n(x)\}$ generates the vector space $T_x(M)$ for all $x \in M$ (Lemma (3.3)). Let $N = \mathbf{R}$.

THEOREM 2. *Let M be a principal G-bundle with fibration $\{M, \pi, B\}$, ξ: $M \to \mathbf{R}^l$ a generator for $C^\infty(M)^G$ and $\{X^1, \ldots, X^n\}$ a generator for $\Gamma^G(T(M))$. Then*

$$\mathfrak{D}_k(M)^G = \{\xi, (X^\alpha)_{|\alpha| \leq k}\}^* C^\infty(\mathbf{R}^l \times \mathbf{R}^{n^k}),$$

for $k \geq 1$.

$(X^\alpha)_{|\alpha| \leq k}$ denotes the sequence of all X^α with $|\alpha| \leq k$ in lexicographical order and $\{\xi, (X^\alpha)_{|\alpha| \leq k}\}^* C^\infty(\mathbf{R}^l \times \mathbf{R}^{n^k})$ is the set of operators of the form $a(\xi, (X^\alpha)_{|\alpha| \leq k})$, $a \in C^\infty(\mathbf{R}^l \times \mathbf{R}^{n^k})$.

In a somewhat different context, problems of this type were studied by Lie [8], by Tresse [11] and more recently by Kumpera [7].

Thanks are due to Ivan Kupka for some helpful discussion.

2. The verification of Theorem 1. Let $J^k(M, N)$ be the kth jet bundle from M into N with source map α and target map β. If P and Q are smooth manifolds, μ: $P \to M$ a diffeomorphism, ν: $N \to Q$ a smooth map, then $(J^k\mu)^*$: $J^k(M, N) \to J^k(P, N)$ and $(J^k\nu)_*$: $J^k(M, N) \to J^k(M, Q)$ are defined by

$$(J^k\mu)^*(\sigma) = j^k_{\mu^{-1}(\alpha(\sigma))} f \circ \mu \quad \text{and} \quad (J^k\nu)_*(\sigma) = j^k_{\alpha(\sigma)} \nu \circ f,$$

where f represents σ.

The action of G on M lifts to a smooth action on $J^k(M, N)$ by

$$g \cdot \sigma = (J^k\phi_g^{-1})^*(\sigma). \tag{2.1}$$

Let π_k be the canonical projection of $J^{k+1}(M, N)$ onto $J^k(M, N)$ and put $D_k = C^\infty(J^k(M, N), \mathbf{R})$. D_∞ is the inductive limit as $k \to \infty$ of (D_k, π_k^*), where π_k^* is the map from D_k to D_{k+1} given by $\pi_k^* F = F \circ \pi_k$.

The set of smooth nonlinear differential operators from $C^\infty(M, N)$ into $C^\infty(M, \mathbf{R})$ of order $\leq k$, $k \in \{\infty, 1, 2, 3, \ldots\}$ is denoted by $\mathfrak{D}_k$. $\mathfrak{F} \in \mathfrak{D}_k$ iff it factorizes as $F \circ j^k$, for some $F \in D_k$. In this case F is called the symbol of $\mathfrak{F}$ or $F = \operatorname{sym} \mathfrak{F}$. The G-equivariant elements of $\mathfrak{D}_k$ are denoted by $\mathfrak{D}_k^G$, the G-invariant elements of D_k by D_k^G.

(2.2) PROPOSITION. $\mathfrak{F} \in \mathfrak{D}_k^G$ *iff* $\operatorname{sym} \mathfrak{F} \in D_k^G$.

PROOF. Let F be the symbol of $\mathfrak{F}$. If F is G-invariant, $f \in C^\infty(M, N)$, $x \in M$, then

$$\begin{aligned}\mathfrak{F}(g \cdot f)(x) &= F(j^k_x g \cdot f) = (g^{-1} \cdot F)(j^k_{g^{-1} \cdot x} f) = F(j^k_{g^{-1} \cdot x} f) \\ &= (\mathfrak{F} f)(g^{-1} \cdot x) = (g \cdot \mathfrak{F} f)(x).\end{aligned}$$

Conversely, if $\mathcal{F}$ is G-equivariant and $\sigma \in J^k(M, N)$ with $\alpha(\sigma) = x$ is represented by f then

$$g \cdot F(\sigma) = F\left(j^k_{g^{-1} \cdot x} g^{-1} \cdot f\right) = \mathcal{F}(g^{-1} \cdot f)(g^{-1} \cdot x)$$
$$= (g^{-1} \cdot \mathcal{F}f)(g^{-1} \cdot x) = \mathcal{F}f(x) = F(\sigma).$$

(2.3) LEMMA. *Let G be a compact Lie group and M a smooth G-manifold with orbit structure of finite type (see* [9]*), then the induced action on $J^k(M, N)$ is of finite orbit type as well.*

PROOF. If $N = \mathbf{R}^n$ and M is a linear G-space then $J^k(M, N)$ is a linear G-space which is of finite orbit type. In the general case we can assume by the Whitney and Mostov embedding theorems that N is smoothly embedded in $\mathbf{R}^n$ and M is smoothly equivariantly embedded in a Euclidean G-space $\mathbf{R}^m$. It will suffice to show that $J^k(M, N)$ is equivariantly embedded in $J^k(\mathbf{R}^m, \mathbf{R}^n)$. Let $\pi: Z \to M$ be an equivariant tubular neighbourhood of M in $\mathbf{R}^m$. Since Z is an open G-invariant set in $\mathbf{R}^m$, $J^k(Z, \mathbf{R}^n)$ is an open G-submanifold of $J^k(\mathbf{R}^m, \mathbf{R}^n)$, so we need only show that $J^k(M, N)$ is equivariantly embedded in $J^k(Z, \mathbf{R}^n)$.

Let i be the inclusion map of N in $\mathbf{R}^n$. Clearly, $(J^k i)_*: J^k(M, N) \to J^k(M, \mathbf{R}^n)$ and $(J^k\pi)^*: J^k(M, \mathbf{R}^n) \to J^k(Z, \mathbf{R}^n)$ are equivariant embeddings. $(J^k\pi)^* \circ (J^k i)_*$ is the desired equivariant embedding of $J^k(M, N)$ into $J^k(Z, \mathbf{R}^n)$. □

Thanks are due to the referee of an earlier version of this section for shortening my original proof.

By Lemma (2.3), the conditions of Theorem 1 imply that the G-manifold $J^k(M, N)$ is of finite orbit type. By a theorem of G. W. Schwarz [**10**, Theorem 2], there exist $A_1, \ldots, A_i \in D_k^G$, such that $F \in D_k^G$ iff $F = f(A_1, \ldots, A_i)$, for some $f \in C^\infty(\mathbf{R}^i, \mathbf{R})$. Hence, the operators $\mathcal{Q}_l$, $1 < l < i$, may be chosen as those with sym $\mathcal{Q}_l = A_l$.

3. The verification of Theorem 2. In this section, M is a smooth principal G-bundle with corresponding fibration $\{M, \pi, B\}$. $m = \dim M$, $a = \dim G$ and $b = m - a = \dim B$.

(3.1) LEMMA. *There exists an invariant generator $\xi: M \to \mathbf{R}^l$, $l < 2b + 1$, for $C^\infty(M)^G$.*

PROOF. By Whitney's embedding theorem there exists an embedding ζ of B into $\mathbf{R}^p$, $p < 2b + 1$. Since $C^\infty(M)^G = \pi^* C^\infty(B)$, we may choose $\xi = \zeta \circ \pi$. □

The action of G on M lifts in the usual way to $T(M)$ by $g \cdot (x, v) = (g \cdot x, g \cdot v)$, where $x \in M$, $v \in T_x(M)$, $g \cdot x = \phi_g(x)$ and $g \cdot v = d\phi_g(x)v$. As in §1, $\Gamma^G(T(M))$ denotes the $C^\infty(M)^G$-module of G-invariant vector fields on M.

(3.2) LEMMA. *There exists a finite generator for $\Gamma^G(T(M))$.*

Proof. For each $x \in M$, let $H_x(M)$ be the horizontal tangent space to M at x with respect to a given principal connection $\mathcal{P}$ on M and let $V_x(M)$ be the vertical tangent space at x. In $T(M)$ we consider the subbundles $H(M) = \bigcup_{x \in M} H_x(M)$ and $V(M) = \bigcup_{x \in M} V_x(M)$. Since $T(M) = H(M) \oplus V(M)$, we need only show that finite generators exist for $\Gamma^G(H(M))$ and $\Gamma^G(V(M))$, the $C^\infty(M)^G$-modules of G-invariant horizontal and vertical vector fields on M.

First we construct a finite generator for $\Gamma^G(H(M))$. By Whitney's embedding theorem we may assume that B is embedded in $\mathbf{R}^p$, $p \leqslant 2b + 1$. Projecting the canonical basis of $T(\mathbf{R}^p)$ onto $T(B)$ we obtain a generator for $T(B)$. The horizontal liftings with respect to $\mathcal{P}$ of the elements of this generator constitute a generator for $\Gamma^G(H(M))$.

$V(M)$ is a G-subbundle of the G-vector bundle $T(M)$. The action of G on $V(M)$ is given by

$$g \cdot (x, v) = (g \cdot x, g \cdot v), \tag{3.3}$$

where $(x, v) \in M \times V_x(M)$.

Let $\text{Lie}(G)$ be the Lie algebra of G. For $l \in \text{Lie}(G)$, let the vertical vector field $\vec{l}$ on M be defined by

$$\vec{l}(x) = \frac{d}{dt}(e^{tl}(x))\bigg|_{t=0}, \qquad x \in M. \tag{3.4}$$

We define a left G-action on $M \times \text{Lie}(G)$ by

$$g \cdot (x, l) = (g \cdot x, \text{Ad}(g)l) \tag{3.5}$$

where $\text{Ad}(g)$ is the adjoint action of $g \in G$ on $\text{Lie}(G)$. With this action, $M \times \text{Lie}(G)$ is isomorphic, as a G-vector bundle, to $V(M)$. An isomorphism is given by $\omega: M \times \text{Lie}(G) \to V(M)$:

$$\omega(x, l) = (x, \vec{l}(x)). \tag{3.6}$$

We check that ω is G-equivariant:

$$\begin{aligned}\omega(g \cdot (x, l)) &= \omega(g \cdot x, \text{Ad}(g)l) = (g \cdot x, \overrightarrow{\text{Ad}(g)\,l}(g \cdot x)) \\ &= (g \cdot x, g \cdot \vec{l}(x)) = g \cdot \omega(x, l),\end{aligned}$$

for all $g \in G$ and $(x, l) \in M \times \text{Lie}(G)$.

Let $E = M \times^G \text{Lie}(G)$ be the vector bundle over B of fiber type $\text{Lie}(G)$ associated with the principal bundle M and the adjoint action of G on $\text{Lie}(G)$ [**4**, XVI, 16.14.7]. In our case E is the quotient of $M \times \text{Lie}(G)$ by the action defined by (3.5). The invariant vertical vector fields on M are in bijective correspondence to the sections of E, (see e.g. [**6**, Theorem 4.8.1]). Since the $C^\infty(B)$-module of cross sections of E is finitely generated, [**5**, p. 76, Lemma 2], the same is true for the $C^\infty(M)^G$-module $\Gamma^G(V(M))$. This completes the proof.

(3.7) **Lemma.** *$\{X^1, \ldots, X^n\} \subset \Gamma^G(T(M))$ is a generator for $\Gamma^G(T(M))$ iff $\{X^1(x), \ldots, X^n(x)\}$ generates the vector space $T_x(M)$, for all $x \in M$.*

PROOF. Clearly a generator of $\Gamma^G(T(M))$ generates the individual tangent spaces. To prove the converse, let

$$\{X^1, \dots, X^n\} \subset \Gamma^G(T(M)) \tag{3.8}$$

be such $\{X^1(x), \dots, X^n(x)\}$ generates $T_x(M)$, for all $x \in M$. For each $x \in X$ we may choose a subset of (3.8)

$$\{X^{j_1(x)}, \dots, X^{j_m(x)}\} \tag{3.9}$$

which, evaluated at x, is a basis for $T_x(M)$. Being a basis is an open condition so there exists an open neighbourhood 0_x of x such that (3.9), evaluated at any $y \in 0_x$ is a basis for $T_y(M)$. Because $g \cdot X^i(x) = X^i(g \cdot x)$, $x \in M$ and $1 < i < n$, and since the action of G on $T(M)$ preserves linear independence (3.9), evaluated at y is a basis for $T_y(M)$ for all $y \in G \cdot 0_x = \pi^{-1}(U_x)$, where $U_x = \pi(0_x)$.

Let $\{V_\alpha\}_{\alpha \in I}$ be a locally finite refinement of the covering $\{U_x\}_{x \in M}$ of B. By the above construction, for each $\alpha \in I$, we have a subset of m elements of (3.8).

$$\{X^{j_\alpha(1)}, \dots, X^{j_\alpha(m)}\}, \tag{3.10}$$

which, evaluated at any $x \in \pi^{-1}(V_\alpha)$, is a basis for $T_x(M)$. Hence any $Y \in \Gamma^G(T(M))$ is of the form

$$Y(x) = \sum_{i=1}^{m} a_{j_\alpha(i)}(x) X^{j_\alpha(i)}(x), \qquad x \in \pi^{-1}(V_\alpha), \tag{3.11}$$

where $a_{j_\alpha(i)} \in C^\infty(\pi^{-1}(V_\alpha))^G$, $1 \leq i \leq m$, $\alpha \in I$.

Let $\{f_\alpha\}_{\alpha \in I}$ be a partition of unity subordinate to $\{V_\alpha\}_{\alpha \in I}$ with $\operatorname{supp} f_\alpha \subset V_\alpha$, $\forall \alpha \in I$. Then

$$Y = \sum_{\alpha \in I} f_\alpha \circ \pi \sum_{i=1}^{m} a_{j_\alpha(i)} X^{j_\alpha(i)} \tag{3.10}$$

which may be written as

$$Y = \sum_{i=1}^{n} b_i X^i, \tag{3.11}$$

with $b_i \in C^\infty(M)^G$, $1 \leq i < m$, since each b_i is locally a finite sum of functions $(f_\alpha \circ \pi) a_{j_\alpha(i)}$ which are smooth and G-invariant.

PROOF OF THEOREM 2. First we consider the case where M is a trivial principal bundle: $M = V \times G$. Moreover we assume that V is an open set of $\mathbf{R}^b$. For $k \in \{1, 2, 3, \dots\}$ let A_k: $J^k(V \times G) \to J^k_{V \times \{e\}}(V \times G)$ be defined by

$$A_k(\sigma) = g^{-1} \cdot \sigma \quad \text{if } \alpha(\sigma) = (v, g). \tag{3.12}$$

Then $\{J^k(V \times G), A_k, J^k_{V \times \{e\}}(V \times G)\}$ is a fibration of the principal G-bundle $J^k(V \times G)$, so

$$C^\infty(J^k(V \times G))^G = A_k^* C^\infty(J^k_{V \times \{e\}}(V \times G)). \tag{3.13}$$

The mapping B_k, defined by

$$B_k\big(j^k_{(b,e)}f\big) = \big(b, j^k_{(0,e)}(f \circ t_b)\big), \tag{3.14}$$

where $f \in C^\infty(\mathbf{R}^b \times G)$ and t_b is the translation in $\mathbf{R}^b \times G$, given by $t_b(a, h) = (a + b, h)$, with $b \in \mathbf{R}^b$ and $(a, h) \in \mathbf{R}^b \times G$, is a diffeomorphism of $J^k_{\mathbf{R}^b \times \{e\}}(\mathbf{R}^b \times G)$ onto $\mathbf{R}^b \times J^k_{(0,e)}(\mathbf{R}^b \times G)$. The space $J^k_{(0,e)}(\mathbf{R}^b \times G)$ carries a natural linear structure. By choosing a basis it is identified with $\mathbf{R}^N$, $N = \dim J^k_{(0,e)}(\mathbf{R}^b \times G)$ and we may consider B_k as a diffeomorphism of $J^k_{\mathbf{R}^b \times \{e\}}(\mathbf{R}^b \times G)$ onto $\mathbf{R}^b \times \mathbf{R}^N$. From (3.13) we obtain

$$C^\infty(J^k(V \times G))^G = (B_k \circ A_k)^* C^\infty(V \times \mathbf{R}^N). \tag{3.15}$$

The canonical projections of $\mathbf{R}^b \times \mathbf{R}^N$ on its first and second factors are denoted by pr_1 and pr_2. Let $\mathcal{P}_k$: $C^\infty(V \times G) \to C^\infty(V \times G, \mathbf{R}^N)$ be the G-equivariant linear operator defined by

$$\mathcal{P}_k = \mathrm{pr}_2 \circ B_k \circ A_k \circ j^k \tag{3.16}$$

and let π' be the canonical projection of $V \times G$ onto V. From Proposition (2.2) and formula (3.15) it follows that $\mathcal{Q} \in \mathfrak{D}^G_k(V \times G)$ iff there exists some $a \in C^\infty(\mathbf{R}^b \times \mathbf{R}^N)$ such that, for all $f \in C^\infty(V \times G)$,

$$\mathcal{Q}f = a\big(\mathrm{pr}_1 \circ B_k \circ A_k \circ j^k, \mathrm{pr}_2 \circ B_k \circ A_k \circ j^k\big) \tag{3.17}$$

which equals a $(\pi', \mathcal{P}_k f) = a(\pi', \mathcal{P}_k)f$. Hence

$$\mathfrak{D}^G_k(V \times G) = (\pi', \mathcal{P}_k)^* C^\infty(\mathbf{R}^b \times \mathbf{R}^N). \tag{3.18}$$

Let $\mathcal{P}^i_k$, $1 < i < N$, denote the ith component of $\mathcal{P}_k$. Clearly each $\mathcal{P}^i_k$ is a linear G-equivariant differential operator on $C^\infty(V \times G)$.

It is easy to see that there exists an invariant basis

$$\{Y^1, \dots, Y^m\} \tag{3.19}$$

for $T(V \times G)$ ($m = \dim V \times G$). From the linearity of $\mathcal{P}_k$ it follows that each $\mathcal{P}^i_k$, $1 < i < N$, can be written uniquely as

$$\mathcal{P}^i_k = \sum_{|\alpha| < k} a_i^\alpha Y^\alpha, \qquad a_i^\alpha \in C^\infty(V \times G), \tag{3.20}$$

(see e.g. [**12**, Theorem 1.1.2]). It follows from the G-equivariance of $\mathcal{P}^i_k$ and of the operators Y^α, $|\alpha| \leqslant k$, that the coefficients a_i^α, $|\alpha| < k$, $1 < i < N$, are G-invariant. Thus we get the $\mathcal{P}^i_k$ in the form

$$\mathcal{P}^i_k = \sum_{|\alpha| < k} (b_i^\alpha \circ \pi') Y^\alpha, \qquad 1 \leqslant i < N, \tag{3.21}$$

where $b_i^\alpha \in C^\infty(V)$, $|\alpha| < k$, $1 < i < N$.

Substituting (3.21) into (3.18) we obtain that $\mathcal{Q} \in \mathfrak{D}^G_k(V \times G)$ iff there exists some $a \in C^\infty(\mathbf{R}^b \times \mathbf{R}^N)$ such that

$$\mathcal{Q}f = a\Big(\pi', \sum_{|\alpha| < k} (b_1^\alpha \circ \pi') Y^\alpha f, \dots, \sum_{|\alpha| < k} (b_N^\alpha \circ \pi') Y^\alpha f\Big), \tag{3.22}$$

for all $f \in C^\infty(V \times G)$. The right-hand side of (3.22) is just a function of π'

and the operators Y^α, $|\alpha| \leq k$. Conversely, any such function represents an element of $\mathfrak{D}_k^G(V \times G)$. Hence,

$$\mathfrak{D}_k^G(V \times G) = (\pi', (Y^\alpha)_{|\alpha| \leq k})C^\infty(\mathbf{R}^b \times \mathbf{R}^{m^k}). \tag{3.23}$$

Let $\{U_\epsilon\}_{\epsilon \in I}$, where I is some index set, be a locally finite atlas for B such that the principal bundles M_ϵ induced by M over U_ϵ, $\epsilon \in I$, are trivializable. Then for each $\epsilon \in I$ there exists an isomorphism λ_ϵ of M_ϵ onto the product bundle $V_\epsilon \times G$, where $V_\epsilon \subset \mathbf{R}^b$ is the parameter domain of U_ϵ.

We define a bijection Λ_ϵ^k of $\mathfrak{D}_\infty^G(V_\epsilon \times G)$ onto $\mathfrak{D}_\infty^G(M_\epsilon)$ by

$$(\Lambda_\epsilon \mathcal{Q})f = \lambda_\epsilon^*(\mathcal{Q}(f \circ \lambda_\epsilon^{-1})), \qquad f \in C^\infty(M_\epsilon). \tag{3.24}$$

Let π_ϵ' be the canonical projection of $V_\epsilon \times G$ onto V_ϵ and let $\{Y_\epsilon^1, \dots, Y_\epsilon^m\}$ be a G-invariant basis for $T(V_\epsilon \times G)$. From (3.23) and (3.24) we obtain that $\mathfrak{B} \in \mathfrak{D}_k^G(M_\epsilon)$ iff

$$\mathfrak{B} = \Lambda_\epsilon(q(\pi_\epsilon', (Y_\epsilon^\alpha)_{|\alpha| \leq k})), \tag{3.25}$$

for some $q \in C^\infty(\mathbf{R}^l \times \mathbf{R}^{m^k})$. Or

$$\mathfrak{B} = q(\pi_\epsilon' \circ \lambda_\epsilon, ((\Lambda_\epsilon Y_\epsilon)^\alpha)_{|\alpha| \leq k}). \tag{3.26}$$

Let ξ_ϵ: $M_\epsilon \to \mathbf{R}^l$ and $\{X_\epsilon^1, \dots, X_\epsilon^n\}$ be the restrictions of the given generators for $C^\infty(M)^G$ and $\Gamma^G(T(M))$ to M_ϵ. Then

$$\pi_\epsilon' \circ \lambda_\epsilon = d \circ \xi_\epsilon, \tag{3.27}$$

for some $d \in C^\infty(\mathbf{R}^l, \mathbf{R}^b)$. Since $\{X_\epsilon^j(x), \dots, X_\epsilon^j(x)\}$ generates $T_x(M_\epsilon)$, $\forall x \in M_\epsilon$, it follows from Lemma (3.7) that

$$\Lambda_\epsilon Y_\epsilon^i = \sum_{j=1}^n (e_\epsilon^{ij} \circ \xi_\epsilon)X_\epsilon^j, \tag{3.28}$$

where $e_\epsilon^{ij} \in C^\infty(\mathbf{R}^l)$, $1 \leq i \leq m$, $1 \leq j \leq n$. After substitution of (3.27) and (3.28) into (3.26) it is easy to see that we may write this equality as

$$\mathfrak{B} = r(\xi_\epsilon, (X_\epsilon^\alpha)_{|\alpha| \leq k}), \tag{3.29}$$

for some $r \in C^\infty(\mathbf{R}^l, \mathbf{R}^{n^k})$.

Let $\{u_\epsilon\}_{\epsilon \in I}$ be a partition of unity on $\mathfrak{B}$ subordinate to $\{U_\epsilon\}_{\epsilon \in I}$ with $\operatorname{supp}(u_\epsilon) \subset U_\epsilon$, $\epsilon \in I$. For a given $\mathfrak{F} \in \mathfrak{D}_k^G(M)$ let $\mathfrak{F}_\epsilon \in \mathfrak{D}_k^G(M_\epsilon)$ be the restriction of $\mathfrak{F}$ to $C^\infty(M_\epsilon)$. (F_ϵ is defined by $\operatorname{sym}\mathfrak{F}_\epsilon = \operatorname{sym}\mathfrak{F}|_{J^k(M_\epsilon)}$.) By (3.29) there exists a $r_\epsilon \in C^\infty(\mathbf{R}^l \times \mathbf{R}^{n^k})$ such that

$$\mathfrak{F}_\epsilon = r_\epsilon(\xi_\epsilon, (X_\epsilon^\alpha)_{|\alpha| \leq k}). \tag{3.30}$$

So

$$\mathfrak{F} = \sum_{\epsilon \in I} (u_\epsilon \circ \pi)r_\epsilon(\xi_\epsilon, (X_\epsilon^\alpha)_{|\alpha| \leq k}). \tag{3.31}$$

Since

$$(U_\epsilon \circ \pi)r_\epsilon(\xi_\epsilon, (X_\epsilon^\alpha)_{|\alpha| \leq k}) = (U_\epsilon \circ \pi)r_\epsilon(\xi, (X^\alpha)_{|\alpha| \leq k})$$

it follows from (3.31) that

$$\mathfrak{F} = a(\xi, (X^\alpha)_{|\alpha| \leq k}), \tag{3.32}$$

for some $a \in C^\infty(\mathbf{R}^l \times \mathbf{R}^{n^k})$. Conversely, (3.32) implies that $\mathfrak{F} \in \mathfrak{D}_k^G(M)$. This completes the proof.

BIBLIOGRAPHY

1. S. Bochner and D. Montgomery, *Groups of differentiable and real or complex analytic transformations*, Ann. of Math. **46** (1945), 685–694.

2. N. Bourbaki, *Lie groups and Lie algebras*, Part I, Addison-Wesley, Reading, Mass., 1975.

3. R. Delver, *Equivariant differential operators of a Lie group. Transformation groups*, (Proc. Conf., Univ. Newcastle-upon-Tyne, August 1976), edited by C. Kosniowski, Cambridge Univ. Press, London, 1977.

4. J. Dieudonné, *Treatise on analysis*, Vol. III, Pure and Appl. Math., Academic Press, New York, 1972.

5. W. Greub, S. Halperin and R. Vanstone, *Connections, curvature and cohomology*, Vol. 1, Academic Press, New York, 1972.

6. D. Husemoller, *Fibre bundles*, 2nd ed., Graduate Texts in Math., Springer-Verlag, Berlin, 1975.

7. A. Kumpera, *Invariants différentiels d'un pseudogroupe de Lie*. I, J. Differential Geometry **10** (1975), 289–345; II, ibid. **10** (1975), no. 3, 347–416.

8. S. Lie, *Allgemeine Untersuchungen über Differentialgleichungen die eine kontinuierliche, endliche Gruppe gestatten*, Math. Ann. **25** (1885), 77–151; Gesammete Abhandlungen, Bd. VI, Teubner, Leipzig, 1927, 139–223.

9. R. S. Palais, *Slices and equivariant imbeddings*, Seminar in Transformation Groups, Chapter VIII, Ann. of Math. Studies, no. 46, Princeton Univ. Press, Princeton, N.J., 1968.

10. G. W. Schwarz, *Smooth functions invariant under the action of a compact Lie group*, Topology **14** (1975), 63–68.

11. A. Tresse, *Sur les invariants differentiels des groupes continus de transformations*, Thèse, Paris, 1893; Acta Math. **18** (1894), 1–88.

12. V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Prentice-Hall, Englewood Cliffs, N.J., 1974.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO, CANADA M5S–1A1

Current address: Department of Mathematics, Free University, 1007 MC, Amsterdam, The Netherlands