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AN AXIOMATIC PROOF OF STIEFEL'S CONJECTURE

JOHN D. BLANTON AND CLINT MCCRORY

ABSTRACT. Stiefel's combinatorial formula for the Stiefel-Whitney homology classes of a smooth manifold is proved, by verifying that the classes defined by his formula satisfy axioms which characterize the Stiefel-Whitney classes.

1. Introduction. In [2] there were presented axioms for the homology duals to the Stiefel-Whitney classes of smooth manifolds. We show here that the homology classes defined by the combinatorial formula of Stiefel [7, p. 342] satisfy these axioms.

Halperin and Toledo published the first detailed proof of Stiefel's conjecture [5]. Earlier proofs were outlined by Whitney [9] and by Cheeger [3]. A proof for mod 2 homology manifolds, using Steenrod operations, was found by Ravenel and McCrory (unpublished). An axiomatic proof for mod 2 homology manifolds has been given recently by L. Taylor [8], using the method of [2] and a classifying space of Quinn.

Let $\mathfrak{M}$ be the category whose objects are C^∞ separable Hausdorff manifolds (without boundary) and whose morphisms are open embeddings, that is $f\colon M \to N$ is a morphism of $\mathfrak{M}$ if M and N are objects of $\mathfrak{M}$ and f is a diffeomorphism of M onto an open subset of N.

Let $\overline{H}_*$ be the homology functor defined using infinite (but locally finite) chains, either singular or simplicial. $\overline{H}_*(\cdot\,; \mathbf{Z}/2)$ is a contravariant functor on the category $\mathfrak{M}$, since an open embedding $f\colon M \to N$ induces a restriction homomorphism [2]

$$f^*\colon \overline{H}_*(N; \mathbf{Z}/2) \to \overline{H}_*(M; \mathbf{Z}/2).$$

The total Stiefel homology class

$$W'(M) = W_0'(M) + W_1'(M) + \cdots + W_m'(M)$$

where m is the dimension of M, satisfies the following axioms:

(1) For every $M \in \operatorname{Obj}(\mathfrak{M})$ and every integer i, $0 < i < m$, there is a Steifel homology class $W_i'(M) \in \overline{H}_{m-i}(M; \mathbf{Z}/2)$.

(2) If $f\colon M \to N$ is a morphism of $\mathfrak{M}$, then $f^* W'(N) = W'(M)$.

(3) $W'(M \times N) = W'(M) \times W'(N)$.

(4) For every nonnegative integer i there exists a positive even integer

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$m > i$ such that

$$W_i'(P_m(\mathbf{R})) = \binom{m+1}{i} x'^i.$$

Here $P_m(\mathbf{R})$ is the real projective space of dimension m and x'^i is the unique nonzero element in $\overline{H}_{m-i}(P_m(\mathbf{R}); \mathbf{Z}/2)$.

In [2] it is proved that there exists a unique homology class $W'(M)$ for each $M \in \mathrm{Obj}(\mathfrak{M})$ such that the axioms (1)–(4) are satisfied.

Following Halperin and Toledo [5], we let (K, φ) denote a smooth triangulation of M, and let K' denote the first barycentric subdivision of K. An infinite simplicial k-chain on M will mean a formal infinite sum $\Sigma\lambda_\sigma\sigma$ where σ is a k-simplex of K' and $\lambda_\sigma \in \mathbf{Z}/2$. These chains form a complex $C_*(M)$ from which $\overline{H}_*(M; \mathbf{Z}/2)$ is defined.

Stiefel [7] conjectured that the infinite chain $s_k(M)$ which is the sum of all the k-simplexes of K' represents the Stiefel homology class $W_{n-k}'(M)$.

We will see below that the chains $s_k(M)$ are cycles, so their homology classes satisfy axiom (1). (This was proved by Akin [1] and by Halperin and Toledo [5].) Since Halperin and Toledo [6], Milnor, and others have shown that Stiefel's combinatorial classes satisfy axiom (3), we prove only that these classes satisfy axioms (2) and (4).

REMARK. Taylor [8] does not prove axiom (2) (his axiom (A1)) for the combinatorial Steifel-Whitney classes! On the other hand, he shows that axiom (4) can be replaced by simpler axioms (his axioms (A3)–(A6)).

2. Axiom (2) is satisfied. If M is a triangulated PL m-manifold with boundary, let $s_k(M)$ be the sum of all the k-simplexes in the first barycentric subdivision of M.

LEMMA 1 (CF. [1, PROPOSITION 1(b)]). $\partial s_k(M) = s_{k-1}(\partial M)$.

PROOF. Let $\alpha = \langle \hat{\sigma}_0, \ldots, \hat{\sigma}_{k-1} \rangle$ be a $(k-1)$-simplex in the first barycentric subdivision, where $\sigma_0 < \cdots < \sigma_{k-1}$ are simplexes in the given triangulation, and $\hat{\sigma}_i$ is the barycenter of $\hat{\sigma}_i$. The coefficient of α in $s_k(M)$ is the mod 2 Euler number of $\mathrm{Link}(\sigma_{k-1})$ (cf. [1, p. 342]). If $\alpha \subset \mathrm{Int}\, M = M \setminus \partial M$ then $\mathrm{Link}(\sigma_{k-1})$ is a sphere. If $\alpha \subset \partial M$ then $\mathrm{Link}(\sigma_{k-1})$ is a disc. $\square$

Let $W_i'(M) \in \overline{H}_{m-i}(M, \partial M; Z/2)$ be the class of $s_{m-i}(M)$.

PROPOSITION 1 (CF. [1, PROPOSITION 2]). *If f: $M \to N$ is a PL homeomorphism of triangulated PL manifolds, $f_* W_i'(M) = W_i'(N)$.*

PROOF. Let M_f be the mapping cylinder of f. M_f is a PL manifold with $\partial M_f = M \cup N$. The given triangulations of M and N can be extended to a triangulation of M_f. Thus, by the lemma, $s_k(M)$ and $s_k(N)$ are homologous in M_f. Let r: $M_f \to N$ be the canonical homotopy equivalence. Since $r|M = f$, $f_* W_{m-k}'(M) = W_{m-k}'(N)$. $\square$

Therefore, by the Whitehead triangulation theorem, we get a well-defined class $W_i'(M) \in \overline{H}_{m-i}(M, \partial M; \mathbf{Z}/2)$ for any smooth m-manifold M with boundary. Let $W_i(M) \in H^i(M; \mathbf{Z}/2)$ be the Poincaré dual class, and let

$$W(M) = W_0(M) + W_1(M) + \cdots + W_m(M) \in H^*(M; \mathbf{Z}/2).$$

PROPOSITION 2. (1) *If f: $M \to N$ is a diffeomorphism,* $f^* W(N) = W(M)$,

(2) $W(\operatorname{Int} M) = W(M)|\operatorname{Int} M$,

(3) $W(\partial M) = W(M)|\partial M$.

PROOF. (1) follows from Proposition 1, and (3) from the lemma. For (2), embed M in the thickened manifold $M_1 = M \cup (\partial M \times [0, \infty))$. Clearly $W(M) = W(M_1)|M$. Applying (3) to the manifold $((\operatorname{Int} M) \times \{0\}) \cup (M_1 \times (0, 1])$, the conclusion follows. $\square$

By definition of the map f^* in homology, axiom (2) for W' is a corollary of the following theorem.

THEOREM 1. *If f: $M \to N$ is a diffeomorphism of M onto an open subset of N, then $f^* W(N) = W(M)$.*

PROOF. By (1) of Proposition 2 we can assume f is an inclusion. By (2) we can assume M and N have no boundary. Applying (3) to the manifold $(M \times \{0\}) \cup (N \times (0, 1])$, the conclusion follows. $\square$

Properties (1)–(3), the proof of (2), and the proof of the theorem are taken from unpublished notes of John Milnor.

3. Axiom (4) is satisfied. Let

$$\Sigma^m = \left\{(x_1, \ldots, x_{m+1}) \in \mathbf{R}^{m+1} \middle| \, |x_1| + \cdots + |x_{m+1}| = 1\right\},$$

a polyhedral m-sphere. Σ^m has a canonical triangulation whose vertices are the intersections of Σ^m with the coordinate axes. Radial projection of Σ^m onto

$$S^m = \left\{(x_1, \ldots, x_{m+1}) \in \mathbf{R}^{m+1} \middle| \, (x_1)^2 + \cdots + (x_{m+1})^2 = 1\right\}$$

gives a smooth triangulation of the standard m-sphere. Let Π^m be the cell complex obtained from Σ^m by identifying x with $-x$ for all $x \in \Sigma^m$. Then the first barycentric subdivision $\Pi'^m = K$ is a simplicial complex which gives a smooth triangulation (K, ϕ) of real projective m-space $P_m(\mathbf{R})$.

PROPOSITION 3. *$W_i'(P_m(\mathbf{R}))$ is represented by the sum of all the $(m - i)$-simplexes of the triangulation K.*

PROOF. By definition, $W_i'(P_m(\mathbf{R}))$ is represented by the sum of all the $(m - i)$-simplexes of the first barycentric subdivision K'. Although K is not the barycentric subdivision of a triangulation, it is the barycentric subdivision of the regular cell complex Π^m whose cells are simplexes. The arguments of Lemma 1 and Proposition 1 apply without change to show that $s_{m-i}(K)$ is homologous to $s_{m-i}(\Pi^m)$. $\square$

REMARK. This proposition also follows from [**6**, Proposition (i), p. 243].

Each k-cell of the m-sphere Σ^m lies in a $(k + 1)$-dimensional linear subspace of $\mathbf{R}^{m+1}$ defined by the vanishing of $m - k$ coordinates of $\mathbf{R}^{m+1}$. For each such linear subspace $\mathbf{R}_j^{k+1}$, let

$$\Sigma^m \cap \mathbf{R}_j^{k+1} = \Sigma_j^k,$$

and let Π_j^k be the image of Σ_j^k in Π^m. There are

$$\binom{m+1}{k+1} = \binom{m+1}{m-k}$$

such k-dimensional projective subspaces Π_j^k in Π^m.

Let $t_k(\Pi^m)$ be the chain of k-simplexes of K which are *not* in the barycentric subdivision of some Π_j^k.

PROPOSITION 4. *The chain $t_k(\Pi^m)$ is a boundary*, $0 \leq k < m$.

PROOF. We shall show that $t_k(\Pi^m)$ is the sum of an even number of mutually homologous k-cycles. Each simplex of $t_k(\Pi^m)$ is contained in a unique k-dimensional projective subspace Λ of $P_m(\mathbf{R})$. We will see that Λ is a subcomplex of K. Thus if $c(\Lambda)$ is the sum of all the k-simplexes of Λ, then $c(\Lambda)$ is a cycle representing the generator of $H_k(P_m(\mathbf{R}); \mathbf{Z}/2)$. Furthermore, the k-simplexes of Λ all belong to $t_k(\Pi^m)$, so $t_k(\Pi^m)$ is the sum of all the cycles $c(\Lambda)$ determined in this way. Finally we will show that there are an even number of such cycles $c(\Lambda)$.

Let σ be a simplex of $t_k(\Pi^m)$, and let s be one of the two k-simplexes of the barycentric subdivision of Σ^m which correspond to σ. Let L be the $(k+1)$-dimensional linear subspace of $\mathbf{R}^{m+1}$ containing s. The image of $L \cap S^m$, under the canonical map $S^m \to P_m(\mathbf{R})$, is the subspace Λ determined by σ.

For $i = 1, \ldots, m+1$, let $\pm v_i$ be the vertex of Σ^m corresponding to $\pm e_i$, where e_i is the ith standard basis vector of $\mathbf{R}^{m+1}$. Then the barycentric coordinate corresponding to $\pm v_i$ in Σ^m is $\pm x_i|\Sigma^m$, where x_i is the ith coordinate function of $\mathbf{R}^{m+1}$.

Let S be the simplex of Σ^m which carries s. Since s is not in the barycentric subdivision of the k-skeleton of Σ^m, we have $\dim S > k$. Let $I = \{i | \pm v_i$ is a vertex of $S\}$, and define $\varepsilon: I \to \{+1, -1\}$ so that $\varepsilon(i)v_i$ is a vertex of S for each $i \in I$. Now each vertex w of s is the barycenter of some face $T(w)$ of S. Let $w_1, \ldots, w_{k+1}$ be the vertices of s, ordered so that $T(w_i)$ is a face of $T(w_j)$ for $i < j$. Define a partition $J = \{J_0, \ldots, J_{k+1}\}$ of the set $\{1, \ldots, m+1\}$ as follows. Let $J_0 = \{1, \ldots, m+1\} - I$, $J_1 = \{i \in I | \varepsilon(i)v_i$ is a vertex of $T(w_1)\}$, and for $p = 2, \ldots, k+1$, let $J_p = \{i \in I | \varepsilon(i)v_i$ is a vertex of $T(w_p)$ but not of $T(w_{p-1})\}$. A dimension count shows that the subspace L of $\mathbf{R}^{m+1}$ spanned by the vertices of s is given by the equations

$$\begin{cases} x_i = 0, & i \in J_0, \\ \varepsilon(i)x_i = \varepsilon(j)x_j, & i, j \in J_p, p = 1, \ldots, k+1. \end{cases} \tag{$*$}$$

Let $|J_p|$ denote the number of elements of J_p. We have $|J_0| = (m+1) - (\dim S + 1) < m - k$. A partition $J = \{J_0, \ldots, J_{k+1}\}$ of $\{1, \ldots, m+1\}$

with $|J_0| < m - k$ will be called *allowable*. Any allowable partition J of $\{1, \ldots, m+1\}$, together with a function ε: $J_1 \cup \cdots \cup J_p \to \{+1, -1\}$, defines a $(k+1)$-dimensional subspace L of $\mathbf{R}^{m+1}$ by the equations $(*)$. The set $L \cap \Sigma^m$ is a subcomplex of the barycentric subdivision Σ'^m, so the corresponding space $\Lambda \subset P^m(\mathbf{R})$ is a subcomplex of K. Since $|J_0| < m - k$, each simplex of $L \cap \Sigma'^m$ is carried by a simplex S of Σ^m with $\dim S > k$, so all the k-simplexes of Λ belong to $t_k(\Pi^m)$, as desired.

It remains to show that there are an even number of such projective subspaces Λ. For each allowable partition J there is at least one p such that $|J_p| > 1$, since $|J_0| < m - k$. For each allowable J, choose such a p, and choose $i_0 \in J_p$. For each Λ corresponding to the partition J and the function ε, let Λ' be the subspace corresponding to J and the function ε' defined by $\varepsilon'(i) = \varepsilon(i)$ for $i \neq i_0$ and $\varepsilon'(i_0) = -\varepsilon(i_0)$. Then $\Lambda' \neq \Lambda$ and $(\Lambda')' = \Lambda$. The existence of this involution $\Lambda \mapsto \Lambda'$ shows that there are an even number of subspaces Λ corresponding to each allowable partition J, so there are an even number of Λ in all. $\square$

THEOREM 2. $W_i'(P_m(\mathbf{R})) = \binom{m+1}{i} x'^i$.

PROOF. Let $k = m - i$. By Proposition 3, $W_i'(P_m(\mathbf{R}))$ is represented by $s_k(\Pi^m)$, the sum of all the k-simplexes in the barycentric subdivision $\Pi'^m = K$. Let $s_k(\Pi_j^k)$ be the sum of all the simplexes of K in Π_j^k. Since Π_j^k is a k-dimensional projective space, $s_k(\Pi_i^k)$ represents the generator x'^i of $H_k(P_m(\mathbf{R}); \mathbf{Z}/2)$. But

$$s_k(\Pi^m) = \sum_{j=1}^{l} s_k(\Pi_j^k) + t_k(\Pi^m), \qquad l = \binom{m+1}{i},$$

and $t_k(\Pi^m)$ is homologous to zero by Proposition 4. $\square$

Theorem 2 implies axiom (4) for *all* integers $i < m$. Theorem 2 has been proved independently by Goldstein and Turner [**4**].

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Department of Mathematics, St. John Fisher College, Rochester, New York 14618

Department of Mathematics, Brown University, Providence, Rhode Island 02912