AN AXIOMATIC PROOF OF STIEFEL'S CONJECTURE

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ABSTRACT. Stiefel's combinatorial formula for the Stiefel-Whitney homology classes of a smooth manifold is proved, by verifying that the classes defined by his formula satisfy axioms which characterize the Stiefel-Whitney classes.

1. Introduction. In [2] there were presented axioms for the homology duals to the Stiefel-Whitney classes of smooth manifolds. We show here that the homology classes defined by the combinatorial formula of Stiefel [7, p. 342] satisfy these axioms.

Halperin and Toledo published the first detailed proof of Stiefel's conjecture [5]. Earlier proofs were outlined by Whitney [9] and by Cheeger [3]. A proof for mod 2 homology manifolds, using Steenrod operations, was found by Ravenel and McCrory (unpublished). An axiomatic proof for mod 2 homology manifolds has been given recently by L. Taylor [8], using the method of [2] and a classifying space of Quinn.

Let \(\mathcal{M}\) be the category whose objects are \(\mathcal{C}\) separable Hausdorff manifolds (without boundary) and whose morphisms are open embeddings, that is \(f: M \rightarrow N\) is a morphism of \(\mathcal{M}\) if \(M\) and \(N\) are objects of \(\mathcal{M}\) and \(f\) is a diffeomorphism of \(M\) onto an open subset of \(N\).

Let \(\overline{H}_*\) be the homology functor defined using infinite (but locally finite) chains, either singular or simplicial. \(\overline{H}_*(\cdot; \mathbb{Z}/2)\) is a contravariant functor on the category \(\mathcal{M}\), since an open embedding \(f: M \rightarrow N\) induces a restriction homomorphism \(f^*: \overline{H}_*(N; \mathbb{Z}/2) \rightarrow \overline{H}_*(M; \mathbb{Z}/2)\).

The total Stiefel homology class

\[ W'(M) = W_0'(M) + W_1'(M) + \cdots + W_m'(M) \]

where \(m\) is the dimension of \(M\), satisfies the following axioms:

1. For every \(M \in \text{Obj}(\mathcal{M})\) and every integer \(i, 0 < i < m\), there is a Steifel homology class \(W_i'(M) \in \overline{H}_{m-i}(M; \mathbb{Z}/2)\).
2. If \(f: M \rightarrow N\) is a morphism of \(\mathcal{M}\), then \(f^* W'(N) = W'(M)\).
3. \(W'(M \times N) = W'(M) \times W'(N)\).
4. For every nonnegative integer \(i\) there exists a positive even integer
\( m > i \) such that

\[
W'_i(P_m(\mathbb{R})) = \left( \begin{array}{c} m + 1 \\ i \end{array} \right) x''.
\]

Here \( P_m(\mathbb{R}) \) is the real projective space of dimension \( m \) and \( x'' \) is the unique nonzero element in \( \overline{H}_{m-i}(P_m(\mathbb{R}); \mathbb{Z}/2) \).

In [2] it is proved that there exists a unique homology class \( W'(M) \) for each \( M \in \text{Obj}(\mathcal{C}) \) such that the axioms (1)–(4) are satisfied.

Following Halperin and Toledo [5], we let \((K, \varphi)\) denote a smooth triangulation of \( M \), and let \( K' \) denote the first barycentric subdivision of \( K \). An infinite simplicial \( k \)-chain on \( M \) will mean a formal infinite sum \( \sum a_i \sigma_i \) where \( a_i \in \mathbb{Z}/2 \). These chains form a complex \( C_*(M) \) from which \( \overline{H}_*(M; \mathbb{Z}/2) \) is defined.

Stiefel [7] conjectured that the infinite chain \( s_k(M) \) which is the sum of all the \( k \)-simplexes of \( K' \) represents the Stiefel homology class \( W'_k(M) \).

We will see below that the chains \( s_k(M) \) are cycles, so their homology classes satisfy axiom (1). (This was proved by Akin [1] and by Halperin and Toledo [5].) Since Halperin and Toledo [6], Milnor, and others have shown that Stiefel's combinatorial classes satisfy axiom (3), we prove only that these classes satisfy axioms (2) and (4).

**Remark.** Taylor [8] does not prove axiom (2) (his axiom (A1)) for the combinatorial Steifel-Whitney classes! On the other hand, he shows that axiom (4) can be replaced by simpler axioms (his axioms (A3)–(A6)).

### 2. Axiom (2) is satisfied

If \( M \) is a triangulated PL \( m \)-manifold with boundary, let \( s_k(M) \) be the sum of all the \( k \)-simplexes in the first barycentric subdivision of \( M \).

**Lemma 1** (cf. [1, Proposition 1(b)]). \( \partial s_k(M) = s_{k-1}(\partial M) \).

**Proof.** Let \( \alpha = <\bar{\sigma}_0, \ldots, \bar{\sigma}_{k-1}> \) be a \( (k-1) \)-simplex in the first barycentric subdivision, where \( \sigma_0 < \cdots < \sigma_{k-1} \) are simplexes in the given triangulation, and \( \bar{\sigma}_i \) is the barycenter of \( \sigma_i \). The coefficient of \( \alpha \) in \( s_k(M) \) is the mod 2 Euler number of \( \text{Link}(\sigma_{k-1}) \) (cf. [1, p. 342]). If \( \alpha \subset \text{Int} M = M \setminus \partial M \) then \( \text{Link}(\sigma_{k-1}) \) is a sphere. If \( \alpha \subset \partial M \) then \( \text{Link}(\sigma_{k-1}) \) is a disc. \( \square \)

Let \( W'_k(M) \) be the class of \( s_{m-i}(M) \).

**Proposition 1** (cf. [1, Proposition 2]). If \( f: M \rightarrow N \) is a PL homeomorphism of triangulated PL manifolds, \( f_* W'_k(M) = W'_k(N) \).

**Proof.** Let \( M_f \) be the mapping cylinder of \( f \). \( M_f \) is a PL manifold with \( \partial M_f = M \cup N \). The given triangulations of \( M \) and \( N \) can be extended to a triangulation of \( M_f \). Thus, by the lemma, \( s_k(M) \) and \( s_k(N) \) are homologous in \( M_f \). Let \( r: M_f \rightarrow N \) be the canonical homotopy equivalence. Since \( r|M = f \), \( f_* W'_{m-k}(M) = W'_{m-k}(N) \). \( \square \)

Therefore, by the Whitehead triangulation theorem, we get a well-defined class \( W'_i(M) \) for any smooth \( m \)-manifold \( M \) with
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Let \( W_i(M) \in H^*(M; \mathbb{Z}/2) \) be the Poincaré dual class, and let
\[
W(M) = W_0(M) + W_1(M) + \cdots + W_m(M) \in H^*(M; \mathbb{Z}/2).
\]

**Proposition 2.** (1) If \( f: M \to N \) is a diffeomorphism, \( f^* W(N) = W(M) \),
(2) \( W(\text{Int } M) = W(M) \mid \text{Int } M \),
(3) \( W(\partial M) = W(M) \mid \partial M \).

**Proof.** (1) follows from Proposition 1, and (3) from the lemma. For (2),
embed \( M \) in the thickened manifold \( M_1 = M \cup (\partial M \times [0, \infty)) \). Clearly
\( W(M) = W(M_1) \mid M \). Applying (3) to the manifold \( ((\text{Int } M) \times \{0\}) \cup (M_1 \times (0, 1]) \), the conclusion follows. □

By definition of the map \( f^* \) in homology, axiom (2) for \( W \) is a corollary of
the following theorem.

**Theorem 1.** If \( f: M \to N \) is a diffeomorphism of \( M \) onto an open subset of \( N \),
then \( f^* W(N) = W(M) \).

**Proof.** By (1) of Proposition 2 we can assume \( f \) is an inclusion. By (2) we
can assume \( M \) and \( N \) have no boundary. Applying (3) to the manifold
\( (M \times \{0\}) \cup (N \times (0, 1]) \), the conclusion follows. □

Properties (1)-(3), the proof of (2), and the proof of the theorem are taken
from unpublished notes of John Milnor.

3. Axiom (4) is satisfied. Let
\[
\Sigma^m = \left\{ (x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} \mid |x_1| + \cdots + |x_{m+1}| = 1 \right\},
\]
a polyhedral \( m \)-sphere. \( \Sigma^m \) has a canonical triangulation whose vertices are
the intersections of \( \Sigma^m \) with the coordinate axes. Radial projection of \( \Sigma^m \) onto
\[
S^m = \left\{ (x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} \mid (x_1)^2 + \cdots + (x_{m+1})^2 = 1 \right\}
\]
gives a smooth triangulation of the standard \( m \)-sphere. Let \( \Pi^m \) be the cell
complex obtained from \( \Sigma^m \) by identifying \( x \) with \( -x \) for all \( x \in \Sigma^m \). Then the
first barycentric subdivision \( \Pi^{m'} = K \) is a simplicial complex which gives a
smooth triangulation \( (K, \phi) \) of real projective \( m \)-space \( P_m(\mathbb{R}) \).

**Proposition 3.** \( W'_i(P_m(\mathbb{R})) \) is represented by the sum of all the \((m-i)\)-simplexes of the triangulation \( K \).

**Proof.** By definition, \( W'_i(P_m(\mathbb{R})) \) is represented by the sum of all the
\((m-i)\)-simplexes of the first barycentric subdivision \( K' \). Although \( K \) is not
the barycentric subdivision of a triangulation, it is the barycentric subdivision
of the regular cell complex \( \Pi^m \) whose cells are simplexes. The arguments of
Lemma 1 and Proposition 1 apply without change to show that \( s_{m-i}(K) \) is
homologous to \( s_{m-i}(\Pi^m) \). □

**Remark.** This proposition also follows from [6, Proposition (i), p. 243].

Each \( k \)-cell of the \( \Sigma \)-sphere \( \Sigma^m \) lies in a \((k+1)\)-dimensional linear subspace of \( \mathbb{R}^{m+1} \) defined by the vanishing of \( m-k \) coordinates of \( \mathbb{R}^{m+1} \). For
each such linear subspace $\mathbb{R}^{k+1}_j$, let

$$
\Sigma^m \cap \mathbb{R}^{k+1}_j = \Sigma^k_j,
$$

and let $\Pi^k_j$ be the image of $\Sigma^k_j$ in $\Pi^m$. There are

$$
\binom{m+1}{k+1} = \binom{m+1}{m-k}
$$

such $k$-dimensional projective subspaces $\Pi^k_j$ in $\Pi^m$.

Let $t_k(\Pi^m)$ be the chain of $k$-simplexes of $K$ which are not in the barycentric subdivision of some $\Pi^k_j$.

**Proposition 4.** The chain $t_k(\Pi^m)$ is a boundary, $0 < k < m$.

**Proof.** We shall show that $t_k(\Pi^m)$ is the sum of an even number of mutually homologous $k$-cycles. Each simplex of $t_k(\Pi^m)$ is contained in a unique $k$-dimensional projective subspace $\Lambda$ of $P_m(\mathbb{R})$. We will see that $\Lambda$ is a subcomplex of $K$. Thus if $c(\Lambda)$ is the sum of all the $k$-simplexes of $\Lambda$, then $c(\Lambda)$ is a cycle representing the generator of $H_k(P_m(\mathbb{R}); \mathbb{Z}/2)$. Furthermore, the $k$-simplexes of $\Lambda$ all belong to $t_k(\Pi^m)$, so $t_k(\Pi^m)$ is the sum of all the cycles $c(\Lambda)$ determined in this way. Finally we will show that there are an even number of such cycles $c(\Lambda)$.

Let $\sigma$ be a simplex of $t_k(\Pi^m)$, and let $s$ be one of the two $k$-simplexes of the barycentric subdivision of $2m$ which correspond to $\sigma$. Let $L$ be the $(k+1)$-dimensional linear subspace of $\mathbb{R}^{m+1}$ containing $s$. The image of $L \cap S^m$, under the canonical map $S^m \to P_m(\mathbb{R})$, is the subspace $\Lambda$ determined by $\sigma$.

For $i = 1, \ldots, m + 1$, let $\pm v_i$ be the vertex of $\Sigma^m$ corresponding to $\pm e_i$, where $e_i$ is the $i$th standard basis vector of $\mathbb{R}^{m+1}$. Then the barycentric coordinate corresponding to $\pm v_i$ in $\Sigma^m$ is $\pm x_i |\Sigma^m|$, where $x_i$ is the $i$th coordinate function of $\mathbb{R}^{m+1}$.

Let $S$ be the simplex of $\Sigma^m$ which carries $s$. Since $s$ is not in the barycentric subdivision of the $k$-skeleton of $\Sigma^m$, we have $\dim S > k$. Let $I = \{i \mid \pm v_i$ is a vertex of $S\}$, and define $\epsilon: I \to \{+1, -1\}$ so that $\epsilon(i)v_i$ is a vertex of $S$ for each $i \in I$. Now each vertex $w$ of $s$ is the barycenter of some face $T(w)$ of $S$. Let $w_1, \ldots, w_{k+1}$ be the vertices of $s$, ordered so that $T(w_i)$ is a face of $T(w_j)$ for $i < j$. Define a partition $J = \{J_0, \ldots, J_{k+1}\}$ of the set $\{1, \ldots, m + 1\}$ as follows. Let $J_0 = \{1, \ldots, m + 1\} - I, J_1 = \{i \in I | \epsilon(i)v_i$ is a vertex of $T(w_1)\}$, and for $p = 2, \ldots, k + 1$, let $J_p = \{i \in I | \epsilon(i)v_i$ is a vertex of $T(w_p)\}$ but not of $T(w_{p-1})\}$. A dimension count shows that the subspace $L$ of $\mathbb{R}^{m+1}$ spanned by the vertices of $s$ is given by the equations

$$
\begin{cases}
  x_i = 0, & i \in J_0, \\
  \epsilon(i)x_i = \epsilon(j)x_j, & i, j \in J_p, p = 1, \ldots, k + 1.
\end{cases}
$$

Let $|J_p|$ denote the number of elements of $J_p$. We have $|J_0| = (m + 1) - (\dim S + 1) < m - k$. A partition $J = \{J_0, \ldots, J_{k+1}\}$ of $\{1, \ldots, m + 1\}$
with \(|J_0| < m - k\) will be called allowable. Any allowable partition \(J\) of \(\{1, \ldots, m + 1\}\), together with a function \(\varepsilon: J_1 \cup \cdots \cup J_p \to \{+1, -1\}\), defines a \((k + 1)\)-dimensional subspace \(L\) of \(\mathbb{R}^{m+1}\) by the equations (*). The set \(L \cap \Sigma^m\) is a subcomplex of the barycentric subdivision \(\Sigma^m\), so the corresponding space \(\Lambda \subset P^m(\mathbb{R})\) is a subcomplex of \(K\). Since \(|J_0| < m - k\), each simplex of \(L \cap \Sigma^m\) is carried by a simplex \(S\) of \(\Sigma^m\) with \(\dim S > k\), so all the \(k\)-simplexes of \(\Lambda\) belong to \(t_k(\Pi^m)\), as desired.

It remains to show that there are an even number of such projective subspaces \(\Lambda\). For each allowable partition \(J\) there is at least one \(p\) such that \(|J_p| > 1\), since \(|J_0| < m - k\). For each allowable \(J\), choose such a \(p\), and choose \(i_0 \in J_p\). For each \(\Lambda\) corresponding to the partition \(J\) and the function \(\varepsilon\), let \(\Lambda'\) be the subspace corresponding to \(J\) and the function \(\varepsilon'\) defined by \(\varepsilon'(i) = \varepsilon(i)\) for \(i \neq i_0\) and \(\varepsilon'(i_0) = -\varepsilon(i_0)\). Then \(\Lambda' \neq \Lambda\) and \((\Lambda')' = \Lambda\). The existence of this involution \(\Lambda \mapsto \Lambda'\) shows that there are an even number of subspaces \(\Lambda\) corresponding to each allowable partition \(J\), so there are an even number of \(\Lambda\) in all. \(\square\)

**Theorem 2.** \(W_i'(P_m(\mathbb{R})) = \binom{m+1}{i} x'^i\).

**Proof.** Let \(k = m - i\). By Proposition 3, \(W_i'(P_m(\mathbb{R}))\) is represented by \(s_k(\Pi^m)\), the sum of all the \(k\)-simplexes in the barycentric subdivision \(\Pi^m\). Let \(s_k(\Pi^i)\) be the sum of all the simplexes of \(K\) in \(\Pi^i\). Since \(\Pi^i\) is a \(k\)-dimensional projective space, \(s_k(\Pi^i)\) represents the generator \(x'^i\) of \(H_k(P_m(\mathbb{R}); \mathbb{Z}/2)\). But

\[
s_k(\Pi^m) = \sum_{j=1}^{l} s_k(\Pi^j) + t_k(\Pi^m), \quad l = \binom{m+1}{i},
\]

and \(t_k(\Pi^m)\) is homologous to zero by Proposition 4. \(\square\)

Theorem 2 implies axiom (4) for all integers \(i < m\). Theorem 2 has been proved independently by Goldstein and Turner [4].

**References**


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