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THE HAHN-BANACH THEOREM IMPLIES SINE'S MEAN ERGODIC THEOREM

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ABSTRACT. Using the Hahn-Banach theorem, a simple proof of Sine's mean ergodic theorem is given.

We shall prove the following.

THEOREM. *Let T be a linear contraction on a Banach space B. Then the ergodic averages $T_n = (1/n)(I + T + \cdots + T^{n-1})$ converge in the strong operator topology if and only if the fixed points of T separate the fixed points of the adjoint operator T^*.*

PROOF. Letting F be the fixed points of T and N the closed linear hull of the set $\{x - Tx: x \in B\}$, the Hahn-Banach theorem implies that the second condition is equivalent to that $F + N$ is a dense subspace of B. And this condition immediately implies that for all $x \in B$, $T_n x$ converges in norm, as the norm of B is complete. Conversely suppose T_n converges strongly to S. Then $x = Sx + (x - Sx)$, $TSx = Sx$, and $T_n(x - Sx) = T_n x - Sx$ $(n \geqslant 1)$. Hence $Sx \in F$, $\lim_n T_n(x - Sx) = 0$, and, for every invariant x^* in B^*,

$$\langle x - Sx, x^* \rangle = \langle T_n(x - Sx), x^* \rangle = 0.$$

The Hahn-Banach theorem again implies that $x - Sx \in N$. This shows that $B = F + N$, completing the proof.

In conclusion I should remark that the above argument applies in more general settings (cf. [1], [2], [3]).

ADDED IN PROOF. After the author submitted this paper, he learned from M. Lin that a similar argument appears in M. Lin, J. Montgomery and R. Sine, *Change of velocity and ergodicity in flows and in Markov semi-groups*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **39** (1977), 197–211.

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