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THE RESIDUAL FINITENESS OF CERTAIN ONE-RELATOR GROUPS

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ABSTRACT. We prove that the groups $\langle a, b; (a^{-1}b^l a b^m)^t\rangle$, where $l, m, t \in Z$ and $t > 2$ are residually finite ($_RF$), thus establishing a conjecture of G. Baumslag [Bull. Amer. Math. Soc. 73 (1967), 618–620].

1. Preliminaries.

LEMMA 1. *Let* $G = \langle x, y;\ (x^l y^m)^t\rangle$. *Then if* $g \in G \setminus \langle x\rangle$ *there exists a homomorphism* τ *of* G *onto a finite group* R *such that* $g\tau \notin \langle x\tau\rangle$ *in* R.

PROOF. Let $T = \langle y, z;\ z^t\rangle$. Then T is LERF [5, p. 359]. Putting $u = zy^{-m}$ we get $T = \langle y, uy^m;\ (uy^m)^t\rangle = \langle y, u;\ (uy^m)^t\rangle$. Now form the generalized free product (g.f.p.)

$$\langle x\rangle \underset{x^l = u}{*} \langle u, y;\ (uy^m)^t\rangle = \left\langle x, y;\ (x^l y^m)^t\right\rangle = G.$$

Suppose $g \in G \setminus \langle x\rangle$. Then g can be expressed as a product $(y_1)x^{i_1}\cdots x^{i_r}(y_{r+1})$ where each $x^{i_k} \notin \langle x^l\rangle$ and each $y_k \in T \setminus \langle x^l\rangle$. Since T is LERF we can find for each y_k a normal subgroup N_k of finite index in T such that $y_k N_k \notin \langle x^l N_k\rangle$. The intersection N of all these N_k is another normal subgroup of finite index in T such that $y_k N \notin \langle x^l N\rangle$ for all (the finitely many) k.

Form the generalized free product

$$S = \langle x\rangle / (\langle x\rangle \cap N) \underset{\langle x^l\rangle/(\langle x^l\rangle \cap N)}{*} T/N.$$

Letting bars denote images under the natural map from G onto S we see by a "form of word" argument that $\bar{g} \notin \langle\bar{x}\rangle$ in S. But S is residually finite [3, p. 194] and so one easily finds a finite homomorphic image R of S in which the image $\bar{g}$ lies outside the image of $\langle\bar{x}\rangle$.

Clearly this result extends easily to

COROLLARY 2. *Let* $G = \langle x, y;\ (x^l y^m)^t\rangle$. *Given* $g_1, \ldots, g_r \in G \setminus \langle x\rangle$ *and* $h_1, \ldots, h_s \in G \setminus \langle y\rangle$ *there exists a normal subgroup* N^* *of finite index in* G *such that* $g_i\langle x\rangle \cap N^* = h_j\langle y\rangle \cap N^* = \varnothing$.

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As a corollary of this we have

COROLLARY 3. $G \in {}_RF$.

PROOF. Let $1 \neq g \in G$. If $g \in \langle y \rangle$ then $g \notin \langle x \rangle$ and so there exists N^* as in Corollary 2 such that $g\langle x \rangle \cap N^* = \varnothing$. In particular g is not in N^*. A similar proof holds if $g \in G \setminus \langle y \rangle$ (whether g is in $\langle x \rangle$ or not).

LEMMA 4. *Let $G = \langle x, y;\ (x^l y^m)^t \rangle$. Then given $r \in Z$, where r is any multiple of l and of m, there exists $N \lhd G$ such that $|G : N| < \infty$, $N \cap \langle x \rangle = \langle x^r \rangle$ and $N \cap \langle y \rangle = \langle y^r \rangle$.*

PROOF. Consider the group $H = \langle g, h;\ (gh)^t, g^\alpha, h^\beta \rangle$ where $r = \alpha l = \beta m$. Then $H \in {}_RF$ [**4**, p. 425]. Thus H has, as a homomorphic image, the finite group $K = \langle g, h;\ (gh)^t, g^\alpha, h^\beta, w_i(g, h) \rangle$ in which g, h have orders α, β exactly.

Now form the g.f.p.

$$\langle x; x^{l\alpha} \rangle \underset{x^l = g}{*} K = \left\langle x, h;\ (x^l h)^t, x^{\alpha l}, h^\beta, w_i(x^l, h) \right\rangle = L,$$

say. Now $L \in {}_RF$ (being a g.f.p. of finite groups) and so has a finite homomorphic image

$$M = \left\langle x, h;\ (x^l h)^t, x^{\alpha l}, h^\beta, w_i(x^l, h), w_i'(x, h) \right\rangle,$$

say, in which x has order αl and h has order β. Now form

$$U = M \underset{h = y^m}{*} \langle y; y^{m\beta} \rangle = \left\langle x, y;\ (x^l y^m)^t, x^r, y^r, w_i(x^l, y^m), w_i'(x, y^m) \right\rangle.$$

This too has a finite homomorphic image

$$V = \left\langle x, y;\ (x^l y^m)^t, x^r, y^r, w_i(x^l, y^m), w_i'(x, y^m), w_i''(x, y) \right\rangle$$

in which x and y both have order r, exactly. Clearly V is a homomorphic image (under ψ, say) of G and if we set $N = \ker \psi\colon G \to V$ we see that $N \cap \langle x \rangle = \langle x^r \rangle$, $N \cap \langle y \rangle = \langle y^r \rangle$ as required.

2. The main theorem.[2]

THEOREM 5. *Let $l, m, t \in Z$ with $t \geqslant 2$. Then $\langle a, b;\ (a^{-1} b^l a b^m)^t \rangle \in {}_RF$.*

PROOF. The given group is well known to be an HNN extension with base group the ${}_RF$ group $B = \langle b_0, b_1;\ (b_1^l b_0^m)^t \rangle$. Further the action of "a" on this group is to conjugate b_1 onto b_0. Thus there is an isomorphism ϕ from $\langle b_1 \rangle$ onto $\langle b_0 \rangle$ coinciding with this action. Thus all the conditions set out in 4.1 of [**1**] are satisfied, their "A", "H", "K" being our B, $\langle b_1 \rangle$, $\langle b_0 \rangle$, respectively. Corollary 2 shows that condition 4.1(a) of [**1**] holds and if N^* does not satisfy

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[2]This theorem was also proved by B. Baumslag and F. Levin several months earlier. The proof given here was done completely independently using an entirely different approach. It is much shorter and perhaps a little crisper. In a communication with G. Baumslag he mentioned that he has also obtained a similar result with a more involved proof.

condition 4.1(b) immediately one readily amends N^* by intersecting it with a suitable N as given by Lemma 4.

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