REFLEXIVE LATTICES OF SUBSPACES[1]

K. H. KIM AND F. W. ROUSH

ABSTRACT. A lattice of subspaces of a vector space is called reflexive if no other subspace is invariant under all linear mappings which leave invariant all subspaces of the lattice. We characterize finite reflexive lattices of subspaces of a finite dimensional vector space over an infinite field.

The subspaces of a vector space form a lattice under the operations $\wedge = \cap$, $\vee =$ span of two subspaces. Any sublattice of this lattice will be called a *lattice of subspaces*.

DEFINITION 1. A lattice L of subspaces of a vector space is *reflexive* if whenever $W \notin L$ there exists a linear transformation A such that $AW \not\subset W$ but $AV \subset V$ for all $V \in L$.

A lattice of subspaces of a vector space will be reflexive if and only if it is lattice of submodules of an algebra $A' \subset A = \text{End } V$.

P. R. Halmos ([2], [3]) has proposed the problem of characterizing reflexive lattices of subspaces. In this paper we solve this problem for finite lattices of subspaces of a finite dimensional vector space, over an infinite field.

DEFINITION 2. A lattice is *distributive* if and only if the identities $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ and $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ are always valid.

A lattice is distributive if and only if it is isomorphic to a lattice of sets.

LEMMA 1. *Let L be a lattice of subsets of $N = \{1, 2, \ldots, n\}$ containing $\varnothing$ and N. Let W be a set not belonging to L. Then there exists a transformation f such that $f(S) \subset S$ for $S \in L$ but $f(W) \not\subset W$.*

PROOF. The lattice L gives a topology on N, if we consider the members of L to be the closed sets of the topology. Since W is not in L, it is not closed. Therefore there exists $u \in W$ such that $\bar{u}$ is not contained in W. Let $v \in \bar{u} \setminus W$. Now define f by $f(x) = x$, $x \neq u$ and $f(u) = v$. Since $f(u) = v$, $f(W) \not\subset W$. Let $S \in L$. If $u \notin S$, then $f(S) = S$ since f will be the identity on S. If $u \in S$ then $f(S) \subset S \cup f\{f(u)\} = S \cup \{v\}$. Since S is closed, $\bar{u} \subset S$. Therefore $v \in S$. So $S \cup \{v\} = S$. Therefore $f(S) \subset S$. This completes the proof.

THEOREM 2. *Let V be a finite dimensional vector space over an infinite field. Let L be a finite lattice of subspace of V. Then L is reflexive if and only if it is distributive.*

PROOF. Suppose L is distributive. Let $\{C_i\}$ be a lattice basis for L, i.e. a minimal set of members of L which together with $\{0\}$ generate the lattice under join. Every finite lattice has a unique basis, namely the join-irreducible elements.

For a basis element C_i, let $\hat{C}_i$ be the span of all members of L which are proper subspaces of C_i. Then $\hat{C}_i$ is a subspace of C_i. Choose subspaces V_i such that $C_i = \hat{C}_i \oplus V_i$. Let $w \in V_k \cap \Sigma_{i \neq k} V_i$. Then

$$w \in C_k \cap \sum_{i \neq k} C_i = \sum C_i \cap C_k \subset \hat{C}_k.$$

Since $w \in V_k \cap \hat{C}_k$, $w = 0$. This proves that ΣV_i is a direct sum.

We next assert that each subspace C_i is the direct sum of those V_k contained in C_i. Suppose not. Let C_j be a minimal member of $\{C_i\}$ which is not spanned by subspaces $V_k \subset C_j$. Then $\hat{C}_j$ is a join of proper subspaces $C_s \subset C_j$ so $\hat{C}_j$ is spanned by V_t: $V_t \subset \hat{C}_j$. But $C_j = \hat{C}_j \oplus V_j$. This is a contradiction, and proves the assertion.

Now let Y be a subspace of V which does not belong to L. To show that L is reflexive, it will suffice to show that there exists a linear mapping $V \to V$ under which each member of L is invariant, but under which Y is not invariant.

If Y is a direct sum of subspaces V_t, then the existence of such a linear mapping follows from Lemma 1. Let π_i denote the projection $V = \bigoplus V_j \to V_i$. If for some i, $\pi_i(Y) \not\subset Y$ let π_i be the mapping. So assume $\pi_i(Y) \subset Y$ for each i. If $0 \neq \pi_i(Y) \neq V_i$ for some i, choose a linear mapping which is the identity on all V_j, $j \neq i$, and which sends V_i into V_i but does not preserve $\pi_i(Y)$. So we may assume $\pi_i(Y) = V_i$ or $\pi_i(Y) = 0$ for each i, and $\pi_i(Y) \subset Y$. Thus if $y \in Y$ has a nontrivial component in V_i, then all of V_i lies in Y. So Y is a direct sum of subspaces V_t, and we are in a case previously dealt with. This proves that a finite distributive lattice of subspaces is reflexive. The converse follows from [4, Lemma 1.2].

REFERENCES

1. G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Publ., no. 25, Amer. Math. Soc., Providence, R. I., 1967.
2. P. R. Halmos, *Finite dimensional Hilbert spaces*, Amer. Math. Monthly 77 (1970), 457–464.
3. _____, *Reflexive lattices of subspaces*, J. London Math. Soc. (2) 4 (1971), 257–263.
4. J. P. Jans, *On the indecomposable representations of algebras*, Ann. of Math. (2) 66 (1957), 418–429.

MATHEMATICS RESEARCH GROUP, ALABAMA STATE UNIVERSITY, MONTGOMERY, ALABAMA 36101