AN ABSTRACT BOREL DENSITY THEOREM

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Abstract. In this paper an abstract form of the Borel density theorem and related results is given centering around the notion of the author's of a (finite dimensional) “admissible” representation. A representation $\rho$ is strongly admissible if each $A'\rho$ is admissible. Although this notion is somewhat technical it is satisfied for certain pairs $(G, \rho)$; e.g., if $G$ is minimally almost periodic and $\rho$ arbitrary, if $G$ is complex analytic and $\rho$ holomorphic. If $G$ is real analytic with radical $R$, $G/R$ has no compact factors and $R$ acts under $\rho$ with real eigenvalues, then $\rho$ is strongly admissible. If in addition $G$ is algebraic/R, then each $R$-rational representation is admissible. The results are proven in three stages where $V$ is defined either over $R$ or $C$.

If $\rho$ is a strongly admissible representation of $G$ on $V$, then each $G$-invariant measure $\mu$ on $\mathfrak{g}(V)$, the Grassmann space of $V$, has support contained in the $G$-fixed point set.

If $\rho$ is a strongly admissible representation of $G$ on $V$ and $G/H$ has finite volume, then each $H$-invariant subspace of $V$ is $G$-invariant.

If $G$ is an algebraic subgroup of $\text{Gl}(V)$ and each rational representation is admissible, then $H$ is Zariski dense in $G$.

The Borel density theorem [1] states that if $G$ is a semisimple linear algebraic group/R and $H$ is a discrete, or more generally a Euclidean closed subgroup such that $G/H$ has finite volume (or more generally has property $S$) then the algebraic (Zariski) hull $H^Z$ of $H$ equals $G$. In [4] I proved certain generalizations of the Borel density theorem in various forms. Principally this was done for minimally almost periodic groups (Furstenberg’s case [3]), complex analytic linear groups (done independently by a different method by S. P. Wang [5]) and real analytic linear groups $G$ with radical $R$ and with the property that $G/R$ has no compact factors, $R$ acts with real eigenvalues and $H$ is a lattice in $G$. While there was a certain underlying unity to these results the methods seemed, on the surface, to be ad hoc. Relying heavily on [4] we present here an abstract form of the theorem which applies to all these cases simultaneously, gives the new result contained in Theorem D, and which in addition proves a generalization of the last-named result of [4]. Finally, the results are now in a form where they could be directly applied to other situations.

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In what follows, $G$ will be a locally compact group, and a continuous finite dimensional real or complex linear representation of $G$ on $V$, $\mathcal{G}(V)$ denotes the Grassmann space of all subspaces of $V$ and $G \times \mathcal{G}(V) \to \mathcal{G}(V)$ will denote the induced action from $\rho$ on $\mathcal{G}(V)$. If $G \times X \to X$ is any action of $G$ on a space $X$ then $X_{\text{fix}}$ or $X_{\text{fix},G}$ denotes the set of $G$-fixed points while $X_c$ or $X_{c,G}$ denotes the $G$-bounded points of $X$; that is those with compact $G$-orbit closures.

An examination of the results of [4] leads to the following definition. We shall say a representation $\rho$ of $G$ is admissible\(^2\) if there is a family of subgroups $\{H_i : i \in I\}$ which generate $G$ and such that each restriction $\rho_i = \rho|H_i$ has the following properties.

(i) $V_{c,\rho, H_i} = V_{\text{fix}, \rho, H_i}$.

(ii) $H_i$ has no closed subgroup of finite index.

(iii) For each $(\rho_i, H_i)$ invariant subspace $W$ of $V$ either $\rho_i$ acts on $W$ by scalars or else the function $g \mapsto \det(g| W)/\|g| W\|^\text{dim } W$ vanishes at $\infty$ on $\rho(H_i)$, where $\| \|$ is any convenient Banach algebra norm on $\text{End } W$.

We shall say that $\rho$ is strongly admissible if each $r$th-exterior power $\Lambda^r \rho$ acting on $\Lambda^r V$ is admissible for $r = 1, \ldots, \text{dim } V$.

The importance of this notion is illustrated by the following theorem which is a slight modification of (1.11) of [4].

**Theorem A.** If $\rho$ is a strongly admissible representation of $G$ on $V$ then each $G$ invariant measure $\mu$ on $\mathcal{G}(V)$ has supp $\mu \subseteq \mathcal{G}(V)_{\text{fix}}$.

We now give sufficient conditions for representations to be admissible.

**Theorem B.** If

(1) $G$ is minimally almost periodic then any continuous representation $\rho$ is admissible,

(2) $G$ is complex analytic then any holomorphic $\rho$ is admissible.

In particular all such representations are strongly admissible.

(3) Suppose $G$ is a real analytic group with $G/R$ having no compact factors and $\rho$ is a representation with the property that $R$ acts with real eigenvalues then $\rho$ is strongly admissible.

(4) Let $G$ be a real linear algebraic subgroup of $\text{Gl}(V)$ which is Euclidean connected, such that $G/R$ has no compact factors and $R$ acts with real eigenvalues then each $\mathbb{R}$-rational representation $\rho$ is admissible.

The first three statements were proven in [4]. To prove (4) we note that since $\rho$ is an analytic representation: $G \to \text{Gl}(W)$ we know that $\rho(R) = \text{rad of } \rho(G)$ and that $\rho(G)/\rho(R)$ has no compact factors. By (3) it is sufficient to see that $\rho(R)$ acts with only real eigenvalues on $W$. Since $R$ is a connected soluble algebraic group acting with real eigenvalues on $W$, $R$ is simply connected by (3.2)a of [4]. But $\rho$ is $\mathbb{R}$-rational and by Lie’s theorem we may

\(^2\)We take the liberty of using the term admissible even though it is used in other contexts not completely disjoint from the present paper, see e.g. Harish-Chandra.
consider rational characters \( \chi \). Since \( R \) is simply connected \( \chi \) takes real values by (3.2)b of [4].

**Corollary.** The conclusion of Theorem A holds in Cases (1), (2) and (3).

Hereafter, we assume the locally compact group \( G \) has a closed subgroup \( H \) and \( G/H \) has a finite \( G \)-invariant measure. Using Theorem A we have as in [4]:

**Theorem C.** If \( \rho: G \to \text{Gl}(V) \) is strongly admissible then each \( H \) invariant subspace of \( V \) is automatically \( G \)-invariant.

In particular, by Theorem B it follows that the conclusion of Theorem C holds in Cases (1), (2) and (3) of Theorem B.

Via (2.4), (2.5) and (2.6) of [4] one has under the assumptions of Theorem C:

**Corollary.** If \( \rho \) is irreducible so is \( \rho|_H \), the linear span of \( \rho(G) \) equals that of \( \rho(H) \) and the centralizer of \( \rho(H) \) in \( \text{End} V \) equals that of \( \rho(G) \).

**Theorem D.** Let \( G \) be a \( k \)-algebraic subgroup of \( \text{Gl}(V) \) where \( k = R \) or \( C \) and suppose each \( k \)-rational representation, \( \rho \) is admissible. Then \( H \) is Zariski dense in \( G \).

**Proof.** Consider the algebraic subgroup \( H^g \) of \( G \). There exists a \( k \)-rational representation \( \rho \) of \( G \) on \( W \) such that \( H^g = \{ g \in G \text{ which leave stable a line } l \text{ in } W \} \) [2]. In particular, \( l \) is \( H^g \) stable and therefore \( H \) stable. Since \( \Lambda^g \rho \) is a \( k \)-rational representation if \( \rho \) is, the hypothesis implies that each \( \rho \) is strongly admissible. By Theorem C, \( l \) is \( G \)-stable. This means \( H^g = G \).

We see by Theorem D and Theorem B that since \( C \)-rational representations are holomorphic that:

**Corollary.** If \( G \) is minimally almost periodic or a complex analytic linear group or a real analytic linear group as in Theorem B(3) then \( H^g = G \).

The last statement generalizes (3.4) of [4] from lattices to arbitrary cofinite volume subgroups.

In closing we note that using algebraic geometry, a different generalization of the Borel density theorem was presented by S. P. Wang in [6].

**Added in Proof.** It has come to the attention of the author that M. S. Raghunathan has also given a proof of the density theorem in the same simple case. Very recently another variant of the density theorem, using ergodic theory, in which a number of the present author's results (but now for \( S \)-subgroups) are reproven has been given by S. Dani.

**References**


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