THE INFINUM OF SMALL SUBHARMONIC FUNCTIONS

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Abstract. Suppose that $u$ is subharmonic in the plane and that, for some $p > 1$, $\lim_{r \to \infty} B(r)/(\log r)^p = \sigma < \infty$. It is shown that, given $\varepsilon > 0$,

$$A(r) > B(r) - (\sigma + \varepsilon) \Re\{(\log r)^p - (\log r + i\pi)^p\}$$

for $r$ outside an exceptional set $E$, where

$$\lim_{r \to \infty} \frac{1}{(\log r)^p-1} \int_{E \cap [1, r]} \frac{(p-1)(\log t)^{p-2}}{t} \, dt < \frac{\sigma}{\sigma + \varepsilon}.$$

1. Introduction. Let $u(z)$ be subharmonic in the plane and define $B(r) = \max_{|z| = r} u(z)$, $A(r) = \inf_{|z| = r} u(z)$. The purpose of this note is to prove

**Theorem.** Let $p > 1$ be given and suppose that $u(z)$ is subharmonic in the plane and satisfies

$$\lim_{r \to \infty} \frac{B(r)}{(\log r)^p} = \sigma < \infty.$$  \hspace{1cm} (1.1)

Then, given $\varepsilon > 0$,

$$A(r) > B(r) - (\sigma + \varepsilon) \Re\{(\log r)^p - (\log r + i\pi)^p\}$$  \hspace{1cm} (1.2)

for all $r$ outside a set $E$ such that

$$\lim_{r \to \infty} \frac{1}{(\log r)^p-1} \int_{E \cap [1, r]} \frac{(p-1)(\log t)^{p-2}}{t} \, dt < \frac{\sigma}{\sigma + \varepsilon}. \hspace{1cm} (1.3)$$

The term $\Re\{(\log r)^p - (\log r + i\pi)^p\}$ is $\frac{1}{2}\pi^2 p(p-1)(\log r)^{p-2}(1 + o(1))$ when $r$ is large, and in this form (1.2) should be compared with Theorem 4 of [1], together with Barry’s remarks in [1, §7.4]. The inequality is evidently sharp as can be seen from $u(z) = \Re(\log z)^p$ (modified slightly in a disc about 0). The case $p = 1$ in (1.1) is considered separately in §4.

If $u(z)$ is subharmonic in the plane then, from the Riesz representation theorem, there exists a unique nonnegative measure $\mu$ defined on all bounded, Borel-measurable subsets of the plane such that, if $R$ is a given positive number,

$$u(z) = h_R(z) + \int_{|z| < R} \log \left|1 - \frac{z}{\xi}\right| \, d\mu_\xi \hspace{1cm} (1.4)$$

for $|z| < R$. Here $h_R(z)$ is harmonic in $|z| < R$. Actually to obtain (1.4) it is assumed that $u$ is harmonic at 0 but this may be achieved in the usual way by replacing $u$ in a small disc about 0 by the Poisson integral of its boundary

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values on the disc. No loss of generality is entailed since we are concerned with \( u(z) \) only when \( |z| \) is large. In what follows we shall assume, without loss of generality, that \( u(0) = 0 \).

In [2] Barry has put into subharmonic form results derived by Kjellberg ([4, pp. 190–192]) in the case \( u(z) = \log|f(z)| \), where \( f \) is an entire function. Some of these are as follows.

Set \( \mu^*(t) = \mu(|\xi| < t) \) and define

\[
\begin{align*}
\mu_1(z, R) &= \int_{|\xi| < R} \log \left| 1 - \frac{z}{\xi} \right| d\mu_\xi, \\
\mu_2(z, R) &= \int_{|\xi| < R} \log \left| 1 + \frac{z}{|\xi|} \right| d\mu_\xi = \int_0^R \log \left| 1 + \frac{z}{t} \right| d\mu^*(t),
\end{align*}
\]

and

\[
\mu_3(z, R) = \mu(z) - \mu_1(z, R).
\]

Then, with \( B_j(r, R) = \max_{|z| = r} u_j(z, R) \), \( A_j(r, R) = \inf_{|z| = r} u_j(z, R) \), \( j = 1, 2, 3 \),

\[
A_2(r, R) < A_1(r, R) < B_1(r, R) < B_2(r, R).
\]

Also

\[
B_3(r, R) < \frac{4r}{R} B(2R), \quad A_3(r, R) > -\frac{4r}{R} B(2R),
\]

for \( 0 < r < \frac{1}{2} R \). From (1.9) it follows that, for \( u(z) \) satisfying (1.1), \( u_1(z, R) \) converges uniformly to \( u(z) \) on bounded sets as \( R \to \infty \) through a sequence.

Finally we note the subharmonic analogue of Jensen’s Theorem [3, p. 473]:

\[
\int_0^r \log \frac{r}{t} d\mu^*(t) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \, d\theta < B(r).
\]

2. A Lemma. We prove the following Lemma which, though very straightforward, is in fact central to the proof of the Theorem.

**Lemma.** Let \( R_1, R_2 \) and \( R \) be positive numbers satisfying \( R_1 < R_2 < R \). Then

\[
I(R_1, R_2, R) = \int_{R_1}^{R_2} \frac{A_2(t, R) - B_3(t, R)}{t} \, dt > \frac{1}{2} \pi^2 \mu^*(R) \left( 1 + \frac{R_2}{R} \right).
\]

On integrating \( z^{-1} \log(1 - z/t) \) around a semiannulus in the upper half-plane we obtain, for any positive \( t \),

\[
\int_{R_1}^{R_2} \left\{ \log \left| 1 - \frac{s}{t} \right| - \log \left( 1 + \frac{s}{t} \right) \right\} \frac{ds}{s}
\]

\[
= \int_0^\pi \text{Arg} \left( 1 - \frac{R_2}{t} e^{i\theta} \right) \, d\theta - \int_0^\pi \text{Arg} \left( 1 - \frac{R_1}{t} e^{i\theta} \right) \, d\theta.
\]

Integrating both sides with respect to \( \mu^*(t) \) from 0 to \( R \) and inverting the order of integration (which is justified since all three integrands are nonposi-
tive) we obtain

\[ I(R_1, R_2, R) = \int_0^\pi d\theta \int_0^R \text{Arg} \left( 1 - \frac{R_2}{t} e^{i\theta} \right) d\mu^*(t) \]

\[ - \int_0^\pi d\theta \int_0^R \text{Arg} \left( 1 - \frac{R_1}{t} e^{i\theta} \right) d\mu^*(t) \]

\[ = I_2 - I_1. \quad (2.3) \]

Integration by parts yields

\[ \int_0^R \text{Arg} \left( 1 - \frac{R_2}{t} e^{i\theta} \right) d\mu^*(t) \]

\[ = \mu^*(R) \text{Arg} \left( 1 - \frac{R_2}{R} e^{i\theta} \right) - \int_0^R \mu^*(t) \frac{\partial}{\partial t} \text{Arg} \left( 1 - \frac{R_2}{t} e^{i\theta} \right) dt \]

\[ = \mu^*(R) \text{Arg} \left( 1 - \frac{R_2}{R} e^{i\theta} \right) - \int_0^R \mu^*(t) \frac{R_2 \sin \theta}{t^2 + R_2^2 - 2tR_2 \cos \theta} dt \]

and thus

\[ I_2 = \mu^*(R) \int_0^\pi \text{Arg} \left( 1 - \frac{R_2}{R} e^{i\theta} \right) d\theta - \int_0^R \mu^*(t) \log \left| \frac{t + R_2}{t - R_2} \right| dt. \]

There is a similar expression for \( I_1 \).

Now

\[ \text{Arg} \left( 1 - \frac{R_2}{R} e^{i\theta} \right) > -\text{Arcsin} \frac{R_2}{R} > -\frac{\pi}{2} \frac{R_2}{R} \]

and also

\[ \int_0^R \mu^*(t) \frac{\log \left| \frac{t + R_2}{t - R_2} \right|}{t} dt < \mu^*(R) \int_0^\infty t^{-1} \log \left| \frac{t + R_2}{t - R_2} \right| dt \]

\[ = \mu^*(R) \int_0^\infty t^{-1} \log \left| \frac{t + 1}{t - 1} \right| dt \]

\[ = \frac{1}{2} \pi^2 \mu^*(R). \]

(The value of the integral follows on taking limits as \( R_1 \to 0 \) and \( R_2 \to \infty \) in (2.2).) Thus

\[ I_2 > -\frac{1}{2} \pi^2 \mu^*(R) \left( 1 + \frac{R_2}{R} \right). \quad (2.4) \]

On the other hand \( I_1 < 0 \), and the Lemma follows.

3. Proof of the Theorem. From (1.8), (1.9) and the Lemma we deduce that, for \( R_2 < \frac{1}{2} R \),

\[ \int_{R_1}^{R_2} \frac{A(t) - B(t)}{t} dt > -\frac{1}{2} \pi^2 \mu^*(R) \left( 1 + \frac{R_2}{R} \right) - \frac{8R_2}{R} B(2R). \quad (3.1) \]
Thus, with \( \psi(t) = \text{Re}\{(\log t)^p - (\log t + i\eta)^p\} \) and \( R_1 = 1 \),
\[
\int_1^{R_2} \frac{A(t) - B(t) + \sigma \psi(t)}{t} \, dt \\
> \frac{\sigma}{p + 1} \text{Re}\{(\log R_2)^{p+1} - (\log R_2 + i\eta)^{p+1}\} \\
- \frac{1}{2} \pi^2 \mu^*(R) \left(1 + \frac{R_2}{R}\right) - \frac{8R_2}{R} B(2R) + O(1). \tag{3.2}
\]

We choose \( R \) so that the second and third terms of the right-hand side of (3.2) are small. This is done as follows. Given \( \eta > 0 \), we can find arbitrarily large values of \( r \) such that
\[
\int_0^r \frac{\mu^*(t)}{t} \, dt < B(r) < (\sigma + \eta)(\log r)^p.
\]
Suppose that \( \mu^*(t) > p(\sigma + 2\eta)(\log t)^{p-1} \) for \( r' < t < r \). Then
\[
(\sigma + 2\eta)\left\{(\log r)^{p} - (\log r')^p\right\} < \int_{r'}^{r} \frac{\mu^*(t)}{t} \, dt < (\sigma + \eta)(\log r)^p,
\]
from which it follows that \( r' > r^* \), where \( \nu > (\eta/(\sigma + 2\eta))^{1/p} \). Also
\[
B(r') < B(r) < (\sigma + \eta)(\log r)^p < \frac{(\sigma + \eta)(\sigma + 2\eta)}{\eta}(\log r)^p.
\]
It is thus possible to find arbitrarily large values of \( r \) at which
\[
\mu^*(r) < p(\sigma + 2\eta)(\log r)^{p-1} \quad \text{and} \quad B(r) < \left(\frac{\sigma + 2\eta}{\eta}\right)^2(\log r)^p,
\]
and we choose \( R \) so that \( 2R \) is one such value. Then
\[
\mu^*(R) < \mu^*(2R) < p(\sigma + 2\eta + o(1))(\log R)^{p-1} \quad \text{and} \quad B(2R) < c(\log 2R)^p, \tag{3.3}
\]
where \( c = (\sigma + 2\eta)^2/\eta \).

Returning to (3.2) and making use of (3.3) we have, for \( R_2 < \frac{1}{2} R \),
\[
J(R_2) = \int_1^{R_2} \frac{A(t) - B(t) + \sigma \psi(t)}{t} \, dt \\
> \frac{\sigma}{p + 1} \text{Re}\{(\log R_2)^{p+1} - (\log R_2 + i\eta)^{p+1}\} \\
- \frac{1}{2} \pi^2 \sigma(\sigma + 2\eta + o(1))(\log R)^{p-1}\left(1 + \frac{R_2}{R}\right) \\
- \frac{8cR_2}{R} (\log 2R)^p + O(1).
\]

Now set \( R_2 = R^{1-\alpha} \), where \( \alpha > 0 \) is fixed. Then
\[
J(R_2) > \frac{1}{2} \pi^2 \sigma \left\{(\sigma - (\sigma + 2\eta)(1 - \alpha)^{-p} + o(1))(\log R_2)^{p-1} \\
- 8cR_2^{-\alpha/(1-\alpha)}(1 - \alpha)^{-p}(\log R_2)^p(1 + o(1)) + O(1)\right\}.
\]
Since we may take \( \eta > 0 \) and \( \alpha > 0 \) as small as we please we deduce that

\[
\lim_{r \to \infty} \frac{J(r)}{(\log r)^{p-1}} > 0. \quad (3.4)
\]

Suppose now that (1.2) is false for \( r \) in a set \( E \). Then

\[
J(r) \leq -\epsilon \int_{E \cap [1, r]} \frac{\psi(t)}{t} \, dt + \sigma \int_{-E \cap [1, r]} \frac{\psi(t)}{t} \, dt
= - (\sigma + \epsilon) \int_{E \cap [1, r]} \frac{\psi(t)}{t} \, dt + \sigma \int_{1}^{r} \frac{\psi(t)}{t} \, dt.
\]

Also \( \psi(t) = \frac{1}{2} \pi p(p - 1)(\log t)^{p-2}(1 + o(1)) \) as \( t \to \infty \) and thus

\[
\lim_{r \to \infty} \frac{J(r)}{(\log r)^{p-1}} \leq \frac{1}{2} \pi^2 p \lim_{r \to \infty} \left\{ \sigma - (\sigma + \epsilon) \frac{1}{(\log r)^{p-1}} \int_{E \cap [1, r]} (p - 1)(\log t)^{p-2} \frac{dt}{t} \right\}.
\]

Comparing this with (3.4) we deduce that

\[
\lim_{r \to \infty} \frac{1}{(\log r)^{p-1}} \int_{E \cap [1, r]} (p - 1)(\log t)^{p-2} \frac{dt}{t} < \frac{\sigma}{\sigma + \epsilon}.
\]

This completes the proof of the Theorem.

**4. The case** \( p = 1 \). When \( p = 1 \) in (1.1) we have on a sequence of \( r \)

\[
\int_{0}^{\infty} \frac{\mu^*(t)}{t} \, dt \leq B(r) = O(\log r). \quad (4.1)
\]

It follows that \( \mu^*(r) \) is bounded on a sequence and thus bounded (since \( \mu^* \) is nondecreasing), so that in fact (4.1) holds for all large \( r \). We may thus appeal to Theorem 12 of [1] to deduce that, if \( h(r) \) is positive and continuous for \( r > c > 0 \) and such that

\[
\int_{c}^{\infty} \frac{h(t)}{t} \, dt
\]

is divergent, then

\[
A(r) > B(r) - h(r)
\]

for certain arbitrarily large values of \( r \). The same result may be obtained from (3.1).

**References**


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