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# THE INFIMUM OF SMALL SUBHARMONIC FUNCTIONS

P. C. FENTON

ABSTRACT. Suppose that $u$ is subharmonic in the plane and that, for some $p > 1$, $\underline{\lim}_{r\to\infty} B(r)/(\log r)^p = \sigma < \infty$. It is shown that, given $\varepsilon > 0$,

$$A(r) > B(r) - (\sigma + \varepsilon)\mathrm{Re}\{(\log r)^p - (\log r + i\pi)^p\}$$

for $r$ outside an exceptional set $E$, where

$$\underline{\lim}_{r\to\infty} \frac{1}{(\log r)^{p-1}} \int_{E\cap[1,r]} \frac{(p-1)(\log t)^{p-2}}{t}\, dt < \frac{\sigma}{\sigma+\varepsilon}.$$

**1. Introduction.** Let $u(z)$ be subharmonic in the plane and define $B(r) = \max_{|z|=r} u(z)$, $A(r) = \inf_{|z|=r} u(z)$. The purpose of this note is to prove

THEOREM. *Let $p > 1$ be given and suppose that $u(z)$ is subharmonic in the plane and satisfies*

$$\underline{\lim}_{r\to\infty} \frac{B(r)}{(\log r)^p} = \sigma < \infty. \tag{1.1}$$

*Then, given $\varepsilon > 0$,*

$$A(r) > B(r) - (\sigma + \varepsilon)\mathrm{Re}\{(\log r)^p - (\log r + i\pi)^p\} \tag{1.2}$$

*for all $r$ outside a set $E$ such that*

$$\underline{\lim}_{r\to\infty} \frac{1}{(\log r)^{p-1}} \int_{E\cap[1,r]} \frac{(p-1)(\log t)^{p-2}}{t}\, dt < \frac{\sigma}{\sigma+\varepsilon}. \tag{1.3}$$

The term $\mathrm{Re}\{(\log r)^p - (\log r + i\pi)^p\}$ is $\frac{1}{2}\pi^2 p(p-1)(\log r)^{p-2}(1 + o(1))$ when $r$ is large, and in this form (1.2) should be compared with Theorem 4 of [**1**], together with Barry's remarks in [**1**, §7.4]. The inequality is evidently sharp as can be seen from $u(z) = \mathrm{Re}(\mathrm{Log}\, z)^p$ (modified slightly in a disc about 0). The case $p = 1$ in (1.1) is considered separately in §4.

If $u(z)$ is subharmonic in the plane then, from the Riesz representation theorem, there exists a unique nonnegative measure $\mu$ defined on all bounded, Borel-measurable subsets of the plane such that, if $R$ is a given positive number,

$$u(z) = h_R(z) + \int_{|\zeta|<R} \log\left|1 - \frac{z}{\zeta}\right| d\mu_\zeta \tag{1.4}$$

for $|z| < R$. Here $h_R(z)$ is harmonic in $|z| < R$. Actually to obtain (1.4) it is assumed that $u$ is harmonic at 0 but this may be achieved in the usual way by replacing $u$ in a small disc about 0 by the Poisson integral of its boundary

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values on the disc. No loss of generality is entailed since we are concerned with $u(z)$ only when $|z|$ is large. In what follows we shall assume, without loss of generality, that $u(0) = 0$.

In [2] Barry has put into subharmonic form results derived by Kjellberg ([4, pp. 190–192]) in the case $u(z) = \log|f(z)|$, where $f$ is an entire function. Some of these are as follows.

Set $\mu^*(t) = \mu(|\zeta| < t)$ and define

$$u_1(z, R) = \int_{|\zeta|<R} \log\left|1 - \frac{z}{\zeta}\right| d\mu_\zeta, \tag{1.5}$$

$$u_2(z, R) = \int_{|\zeta|<R} \log\left|1 + \frac{z}{|\zeta|}\right| d\mu_\zeta = \int_0^R \log\left|1 + \frac{z}{t}\right| d\mu^*(t), \tag{1.6}$$

and

$$u_3(z, R) = u(z) - u_1(z, R). \tag{1.7}$$

Then, with $B_j(r, R) = \max_{|z|=r} u_j(z, R)$, $A_j(r, R) = \inf_{|z|=r} u_j(z, R)$, $j = 1, 2, 3$,

$$A_2(r, R) \leqslant A_1(r, R) \leqslant B_1(r, R) \leqslant B_2(r, R). \tag{1.8}$$

Also

$$B_3(r, R) \leqslant \frac{4r}{R} B(2R), \qquad A_3(r, R) \geqslant -\frac{4r}{R} B(2R), \tag{1.9}$$

for $0 \leqslant r \leqslant \frac{1}{2}R$. From (1.9) it follows that, for $u(z)$ satisfying (1.1), $u_1(z, R)$ converges uniformly to $u(z)$ on bounded sets as $R \to \infty$ through a sequence.

Finally we note the subharmonic analogue of Jensen's Theorem [3, p. 473]: for $r > 0$,

$$\int_0^r \log \frac{r}{t}\, d\mu^*(t) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\, d\theta \leqslant B(r). \tag{1.10}$$

**2. A Lemma.** We prove the following Lemma which, though very straightforward, is in fact central to the proof of the Theorem.

LEMMA. *Let $R_1$, $R_2$ and $R$ be positive numbers satisfying $R_1 < R_2 < R$. Then*

$$I(R_1, R_2, R) = \int_{R_1}^{R_2} \frac{A_2(t, R) - B_2(t, R)}{t}\, dt \geqslant -\tfrac{1}{2}\pi^2 \mu^*(R)\left(1 + \frac{R_2}{R}\right). \tag{2.1}$$

On integrating $z^{-1}\operatorname{Log}(1 - z/t)$ around a semiannulus in the upper half-plane we obtain, for any positive $t$,

$$\int_{R_1}^{R_2} \left\{\log\left|1 - \frac{s}{t}\right| - \log\left(1 + \frac{s}{t}\right)\right\} \frac{ds}{s}$$
$$= \int_0^\pi \operatorname{Arg}\left(1 - \frac{R_2}{t} e^{i\theta}\right) d\theta - \int_0^\pi \operatorname{Arg}\left(1 - \frac{R_1}{t} e^{i\theta}\right) d\theta. \tag{2.2}$$

Integrating both sides with respect to $\mu^*(t)$ from 0 to $R$ and inverting the order of integration (which is justified since all three integrands are nonposi-

tive) we obtain

$$\begin{aligned} I(R_1, R_2, R) &= \int_0^{\pi} d\theta \int_0^R \operatorname{Arg}\left(1 - \frac{R_2}{t} e^{i\theta}\right) d\mu^*(t) \\ &\quad - \int_0^{\pi} d\theta \int_0^R \operatorname{Arg}\left(1 - \frac{R_1}{t} e^{i\theta}\right) d\mu^*(t) \\ &= I_2 - I_1. \end{aligned} \tag{2.3}$$

Integration by parts yields

$$\begin{aligned} &\int_0^R \operatorname{Arg}\left(1 - \frac{R_2}{t} e^{i\theta}\right) d\mu^*(t) \\ &\quad = \mu^*(R)\operatorname{Arg}\left(1 - \frac{R_2}{R} e^{i\theta}\right) - \int_0^R \mu^*(t) \frac{\partial}{\partial t} \operatorname{Arg}\left(1 - \frac{R_2}{t} e^{i\theta}\right) dt \\ &\quad = \mu^*(R)\operatorname{Arg}\left(1 - \frac{R_2}{R} e^{i\theta}\right) - \int_0^R \mu^*(t) \frac{R_2 \sin\theta}{t^2 + R_2^2 - 2tR_2 \cos\theta} dt \end{aligned}$$

and thus

$$I_2 = \mu^*(R) \int_0^{\pi} \operatorname{Arg}\left(1 - \frac{R_2}{R} e^{i\theta}\right) d\theta - \int_0^R \frac{\mu^*(t)}{t} \log\left|\frac{t + R_2}{t - R_2}\right| dt.$$

There is a similar expression for $I_1$.

Now

$$\operatorname{Arg}\left(1 - \frac{R_2}{R} e^{i\theta}\right) \geqslant -\operatorname{Arc}\sin \frac{R_2}{R} \geqslant -\frac{\pi}{2} \frac{R_2}{R}$$

and also

$$\begin{aligned} \int_0^R \frac{\mu^*(t)}{t} \log\left|\frac{t + R_2}{t - R_2}\right| dt &\leqslant \mu^*(R) \int_0^{\infty} t^{-1} \log\left|\frac{t + R_2}{t - R_2}\right| dt \\ &= \mu^*(R) \int_0^{\infty} t^{-1} \log\left|\frac{t + 1}{t - 1}\right| dt \\ &= \frac{1}{2} \pi^2 \mu^*(R). \end{aligned}$$

(The value of the integral follows on taking limits as $R_1 \to 0$ and $R_2 \to \infty$ in (2.2).) Thus

$$I_2 \geqslant -\frac{1}{2} \pi^2 \mu^*(R)\left(1 + \frac{R_2}{R}\right). \tag{2.4}$$

On the other hand $I_1 \leqslant 0$, and the Lemma follows.

**3. Proof of the Theorem.** From (1.8), (1.9) and the Lemma we deduce that, for $R_2 \leqslant \frac{1}{2}R$,

$$\int_{R_1}^{R_2} \frac{A(t) - B(t)}{t} dt \geqslant -\frac{1}{2} \pi^2 \mu^*(R)\left(1 + \frac{R_2}{R}\right) - \frac{8R_2}{R} B(2R). \tag{3.1}$$

Thus, with $\psi(t) = \mathrm{Re}\{(\log t)^p - (\log t + i\pi)^p\}$ and $R_1 = 1$,

$$\int_1^{R_2} \frac{A(t) - B(t) + \sigma\psi(t)}{t}\, dt$$
$$\geqslant \frac{\sigma}{p+1} \mathrm{Re}\{(\log R_2)^{p+1} - (\log R_2 + i\pi)^{p+1}\}$$
$$- \frac{1}{2}\pi^2\mu^*(R)\left(1 + \frac{R_2}{R}\right) - \frac{8R_2}{R} B(2R) + O(1). \tag{3.2}$$

We choose $R$ so that the second and third terms of the right-hand side of (3.2) are small. This is done as follows. Given $\eta > 0$, we can find arbitrarily large values of $r$ such that

$$\int_0^r \frac{\mu^*(t)}{t}\, dt \leqslant B(r) \leqslant (\sigma + \eta)(\log r)^p.$$

Suppose that $\mu^*(t) > p(\sigma + 2\eta)(\log t)^{p-1}$ for $r' \leqslant t \leqslant r$. Then

$$(\sigma + 2\eta)\{(\log r)^p - (\log r')^p\} \leqslant \int_{r'}^r \frac{\mu^*(t)}{t}\, dt \leqslant (\sigma + \eta)(\log r)^p,$$

from which it follows that $r' \geqslant r^\nu$, where $\nu = (\eta/(\sigma + 2\eta))^{1/p}$. Also

$$B(r') \leqslant B(r) \leqslant (\sigma + \eta)(\log r)^p \leqslant \frac{(\sigma + \eta)(\sigma + 2\eta)}{\eta}(\log r')^p.$$

It is thus possible to find arbitrarily large values of $r$ at which

$$\mu^*(r) \leqslant p(\sigma + 2\eta)(\log r)^{p-1} \quad \text{and} \quad B(r) \leqslant \frac{(\sigma + 2\eta)^2}{\eta}(\log r)^p,$$

and we choose $R$ so that $2R$ is one such value. Then

$$\mu^*(R) \leqslant \mu^*(2R) \leqslant p(\sigma + 2\eta + o(1))(\log R)^{p-1} \quad \text{and}$$
$$B(2R) \leqslant c(\log 2R)^p, \tag{3.3}$$

where $c = (\sigma + 2\eta)^2/\eta$.

Returning to (3.2) and making use of (3.3) we have, for $R_2 \leqslant \frac{1}{2}R$,

$$J(R_2) = \int_1^{R_2} \frac{A(t) - B(t) + \sigma\psi(t)}{t}\, dt$$
$$\geqslant \frac{\sigma}{p+1} \mathrm{Re}\{(\log R_2)^{p+1} - (\log R_2 + i\pi)^{p+1}\}$$
$$- \frac{1}{2}\pi^2 p(\sigma + 2\eta + o(1))(\log R)^{p-1}\left(1 + \frac{R_2}{R}\right)$$
$$- \frac{8cR_2}{R}(\log 2R)^p + O(1).$$

Now set $R_2 = R^{1-\alpha}$, where $\alpha > 0$ is fixed. Then

$$J(R_2) \geqslant \frac{1}{2}\pi^2 p\{\sigma - (\sigma + 2\eta)(1 - \alpha)^{1-p} + o(1)\}(\log R_2)^{p-1}$$
$$- 8cR_2^{-\alpha/(1-\alpha)}(1 - \alpha)^{-p}(\log R_2)^p(1 + o(1)) + O(1).$$

Since we may take $\eta > 0$ and $\alpha > 0$ as small as we please we deduce that

$$\overline{\lim_{r\to\infty}} \frac{J(r)}{(\log r)^{p-1}} \geqslant 0. \tag{3.4}$$

Suppose now that (1.2) is false for $r$ in a set $E$. Then

$$\begin{aligned} J(r) &\leqslant -\varepsilon \int_{E\cap[1,r]} \frac{\psi(t)}{t}\,dt + \sigma \int_{\sim E\cap[1,r]} \frac{\psi(t)}{t}\,dt \\ &= -(\sigma+\varepsilon)\int_{E\cap[1,r]} \frac{\psi(t)}{t}\,dt + \sigma\int_1^r \frac{\psi(t)}{t}\,dt. \end{aligned}$$

Also $\psi(t) = \frac{1}{2}\pi^2 p(p-1)(\log t)^{p-2}(1+o(1))$ as $t\to\infty$ and thus

$$\begin{aligned} &\overline{\lim_{r\to\infty}} \frac{J(r)}{(\log r)^{p-1}} \\ &\leqslant \frac{1}{2}\pi^2 p \,\overline{\lim_{r\to\infty}} \left\{ \sigma - (\sigma+\varepsilon)\frac{1}{(\log r)^{p-1}} \int_{E\cap[1,r]} (p-1)\frac{(\log t)^{p-2}}{t}\,dt \right\}. \end{aligned}$$

Comparing this with (3.4) we deduce that

$$\underline{\lim_{r\to\infty}} \frac{1}{(\log r)^{p-1}} \int_{E\cap[1,r]} (p-1)\frac{(\log t)^{p-2}}{t}\,dt \leqslant \frac{\sigma}{\sigma+\varepsilon}.$$

This completes the proof of the Theorem.

**4. The case** $p = 1$. When $p = 1$ in (1.1) we have on a sequence of $r$

$$\int_0^r \frac{\mu^*(t)}{t}\,dt \leqslant B(r) = O(\log r). \tag{4.1}$$

It follows that $\mu^*(r)$ is bounded on a sequence and thus bounded (since $\mu^*$ is nondecreasing), so that in fact (4.1) holds for all large $r$. We may thus appeal to Theorem 12 of [1] to deduce that, if $h(r)$ is positive and continuous for $r \geqslant c > 0$ and such that

$$\int_c^\infty \frac{h(t)}{t}\,dt$$

is divergent, then

$$A(r) > B(r) - h(r)$$

for certain arbitrarily large values of $r$. The same result may be obtained from (3.1).

## References

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Department of Mathematics, University of Otago, Otago, New Zealand