OSCILLATION OF FIRST-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

YUICHI KITAMURA AND TAKAŠI KUSANO

Abstract. This paper is devoted to the study of the oscillatory behavior of solutions of the first-order nonlinear functional differential equations

\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) \\
&\quad + F(t, x(t), x(g_1(t)), \ldots, x(g_N(t))), \quad (A) \\
\dot{x}(t) + \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) \\
&\quad + F(t, x(t), x(g_1(t)), \ldots, x(g_N(t))) = 0. \quad (B)
\end{align*}

First, without assuming that the deviating arguments \( g_i(t), 1 < i < N, \) are retarded or advanced, sufficient conditions are established for all solutions of (A) and (B) to be oscillatory.

Secondly, a characterization of oscillation of all solutions is obtained for equation (A) with \( F \equiv 0 \) and \( g_i(t) > t, 1 < i < N, \) as well as for equation (B) with \( F \equiv 0 \) and \( g_i(t) < t, 1 < i < N. \)

The purpose of this paper is to obtain oscillation criteria for the first order differential equations

\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) \\
&\quad + F(t, x(t), x(g_1(t)), \ldots, x(g_N(t))), \quad (A) \\
\dot{x}(t) + \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) \\
&\quad + F(t, x(t), x(g_1(t)), \ldots, x(g_N(t))) = 0, \quad (B)
\end{align*}

where the following conditions are assumed to hold:

(a) \( q_i, g_i \in C[[a, \infty), R], q_i(t) > 0, \) and \( \lim_{t \to \infty} g_i(t) = \infty, 1 < i < N; \)
(b) \( f_i \in C[R, R], f_i \) is nondecreasing, and \( u_f(u) > 0 \) for \( u \neq 0, 1 < i < N; \)
(c) \( F \in C[[a, \infty) \times R^{N+1}, R], \) and \( u_0 F(t, u_0, u_1, \ldots, u_N) > 0 \) for \( u_0 u_i > 0, 1 < i < N. \)

In what follows, by a proper solution of (A) or (B), we mean a function \( x \in C^1[[T_0, \infty), R] \) which satisfies (A) or (B) for all sufficiently large \( t \) and \( \sup \{|x(t)|: t > T\} > 0 \) for any \( T > T_0. \) The standing hypothesis is that equations (A) and (B) do possess proper solutions. A proper solution of (A) or

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(B) is called oscillatory if it has arbitrarily large zeros and it is called nonoscillatory otherwise.

The main results of this paper are as follows.

**THEOREM 1.** Suppose that each \( f_i, 1 < i < N, \) satisfies

\[
\int_{-M}^{\infty} \frac{du}{f_i(u)} < \infty \quad \text{and} \quad \int_{-\infty}^{-M} \frac{du}{f_i(u)} < \infty \quad \text{for any } M > 0. \tag{1}
\]

All proper solutions of (A) are oscillatory if

\[
\sum_{i=1}^{N} \int_{\mathcal{A}_i} q_i(t) \, dt = \infty, \tag{2}
\]

where \( \mathcal{A}_i = \{ t \in [a, \infty): g_i(t) > t \}, \) the advanced part of \( g_i(t). \)

**THEOREM 2.** Suppose that each \( f_i, 1 < i < N, \) satisfies

\[
\int_{0}^{m} \frac{du}{f_i(u)} < \infty \quad \text{and} \quad \int_{0}^{-m} \frac{du}{f_i(u)} < \infty \quad \text{for any } m > 0. \tag{3}
\]

All proper solutions of (B) are oscillatory if

\[
\sum_{i=1}^{N} \int_{\mathcal{A}_i} q_i(t) \, dt = \infty, \tag{4}
\]

where \( \mathcal{A}_i = \{ t \in [a, \infty): a < g_i(t) < t \}, \) the retarded part of \( g_i(t). \)

All the literature on the oscillation of first-order functional differential equations has been devoted to the case where the deviating arguments involved are retarded or advanced (see, for example, [1]–[10]), and so the above theorems can be covered by none of the previous results.

**Proof of Theorem 1.** Let \( x(t) \) be a nonoscillatory solution which is eventually positive. There is \( T > a \) such that \( x(t) > 0 \) and \( x(g_i(t)) > 0 \) for \( t > T, \ 1 < i < N. \) By conditions (b) and (c), \( f_j(x(t)) > 0, \ 1 < j < N, \) and \( F(t, x(t), . . . ) > 0 \) on \( [T, \infty), \) and so from (A), \( x'(t) > 0 \) for \( i > T, \) which implies that the \( f_j(x(t)) \) are nondecreasing on \( [T, \infty). \) Let \( i \) be fixed. We divide (A) by \( f_j(x(t)) \) and integrate it on \( [T, T'], \ T' > T. \) Using condition (c) and noting that \( f_j(x(g_i(t))) > f_j(x(t)) \) for \( t \in \mathcal{A}_i \cap [T, T'], \) we then have

\[
\int_{T}^{T'} \frac{x'(t)}{f_j(x(t))} \, dt > \int_{T}^{T'} q_i(t) \frac{f_j(x(g_i(t)))}{f_j(x(t))} \, dt \]

\[
> \int_{\mathcal{A}_i \cap [T, T']} q_i(t) \, dt. \tag{5}
\]

Letting \( T' \to \infty \) in (5) and taking (1) into account, we find

\[
\int_{\mathcal{A}_i \cap [T, \infty)} q_i(t) \, dt < \int_{x(T)}^{x(\infty)} \frac{du}{f_j(u)} < \infty.
\]

Since \( i \) is arbitrary, this contradicts (2), and hence (A) cannot have eventually positive proper solutions. Similarly, (A) does not possess eventually negative proper solutions.
Proof of Theorem 2. Let \( x(t) \) be a nonoscillatory solution of (B). Without loss of generality we may suppose that \( x(t) \) is eventually positive. There is \( t_0 > a \) such that \( x(t) > 0 \) and \( x(g_i(t)) > 0 \) for \( t > t_0, 1 < i < N \). Take \( T > t_0 \) so large that \( g_i(t) > t_0 \) for \( t > T, 1 < i < N \). Since \( x'(t) < 0, t > t_0 \) by (B), the \( f_i(x(t)) \) are positive and nonincreasing on \( [t_0, \infty) \), so that \( f_i(x(g_i(t))) > f_i(x(t)) \) for \( t \in (t_0, T] \). Proceeding as in the proof of Theorem 1, we obtain from (B)

\[
\int_T^{T'} \frac{-x'(t)}{f_i(x(t))} \, dt > \int_T^{T'} q_i(t) \frac{f_i(x(g_i(t)))}{f_i(x(t))} \, dt
\]

\[
> \int_{\mathbb{R}_+ \cap [T, T']} q_i(t) \, dt. \tag{6}
\]

Letting \( T' \to \infty \) in (6) and using (3), we see that

\[
\int_{\mathbb{R}_+ \cap [T, \infty)} q_i(t) \, dt < \int_{x(\infty)}^{x(T)} \frac{du}{f_i(u)} < \infty
\]

for \( 1 < i < N \), which contradicts (4). This completes the proof.

Remark. If \( g_i(t) > t, 1 < i < N \) (resp. \( g_i(t) < t, 1 < i < N \)), then condition (2) (resp. (4)) reduces to

\[
\sum_{i=1}^{N} \int_{\mathbb{R}_+} q_i(t) \, dt < \infty. \tag{7}
\]

Thus Theorem 1 is an extension of a result of Anderson [1, Theorem 3].

We now consider the particular cases of (A) and (B).

\[
x'(t) = \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))), \tag{A_0}
\]

\[
x'(t) + \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) = 0. \tag{B_0}
\]

A sufficient condition for (A_0) and (B_0) to have nonoscillatory solutions is given in the following theorem.

Theorem 3. Let conditions (a) and (b) hold. If

\[
\sum_{i=1}^{N} \int_{\mathbb{R}_+} q_i(t) \, dt < \infty, \tag{8}
\]

then equations (A_0) and (B_0) have nonoscillatory solutions.

Proof. For an arbitrarily given constant \( k > 0 \), consider the integral equation

\[
x(t) = k + \sum_{i=1}^{N} \int_{T}^{t} q_i(s) f_i(x(g_i(s))) \, ds, \tag{9}
\]

where \( T > a \) is chosen so that

\[
\sum_{i=1}^{N} f_i(2k) \int_{T}^{\infty} q_i(s) \, ds < k.
\]
Put $T_0 = \min_{1 \leq i \leq N} \inf_{T \geq T_0} g_i(t)$ and let $C$ denote the locally convex space of all continuous functions $x: [T_0, \infty) \to R$ with the topology of uniform convergence on compact subintervals of $[T_0, \infty)$. Let $X = \{ x \in C: k < x(t) < 2k, t > T_0 \}$. Define the operator $\Phi: X \to C$ by

$$\Phi x(t) = k + \sum_{i=1}^{N} \int_{T}^{t} q_i(s)f_i(x(g_i(s))) \, ds, \quad t > T,$$

$$\Phi x(t) = k, \quad T_0 < t \leq T. \quad (10)$$

It is easy to verify that $\Phi$ maps $X$, which is a closed convex subset of $C$, continuously into a compact subset of $X$. Consequently, by the Tychonoff fixed-point theorem, $\Phi$ has a fixed point $x$ in $X$. Obviously, this fixed point $x = x(t)$ satisfies (9) for $t > T$ and hence becomes a nonoscillatory solution of (A0).

Similarly, a nonoscillatory solution of (B0) is obtained as a solution to the integral equation

$$x(t) = 2k - \sum_{i=1}^{N} \int_{T}^{t} q_i(s)f_i(x(g_i(s))) \, ds.$$  

It would be of interest to observe that by combining Theorems 1 and 2 with Theorem 3 one easily obtains a characterization of oscillation of (A0) in the advanced case and equation (B0) in the retarded case.

**Theorem 4.** Suppose that (1) holds and that $g_i(t) > t, 1 < i < N$. Then (7) is a necessary and sufficient condition for all proper solutions of (A0) to be oscillatory.

**Theorem 5.** Suppose that (3) holds and that $g_i(t) < t, 1 < i < N$. Then (7) is a necessary and sufficient condition for all proper solutions of (B0) to be oscillatory.

**Remark.** Theorem 5 was first proved by Koplatadze [2].

**Example.** Consider the equation

$$x'(t) = \frac{|x(t + \sin t)|^{\alpha} \text{sgn} x(t + \sin t)}{t^{\beta} \left[ \log(t + \sin t) \right]^{\alpha}}, \quad t \geq 2\pi, \quad (11)$$

where $\alpha > 0$ and $\beta$ are real constants. The advanced part of $g(t) = t + \sin t$ is $\mathcal{A} = \cup_{k=1}^{\infty} (2k\pi, (2k + 1)\pi)$.

(i) Let $\alpha > 1$. If $\beta < 1$, then

$$\int_{\mathcal{A}} \frac{dt}{t^{\beta} \left[ \log(t + \sin t) \right]^{\alpha}} = \sum_{k=1}^{\infty} \int_{2k\pi}^{(2k+1)\pi} \frac{dt}{t^{\beta} \left[ \log(t + \sin t) \right]^{\alpha}} = \infty, \quad (12)$$

and so from Theorem 1 it follows that all proper solutions of (11) are oscillatory. If $\beta > 1$, then

$$\int_{2\pi}^{\infty} \frac{dt}{t^{\beta} \left[ \log(t + \sin t) \right]^{\alpha}} < \infty,$$

and hence, by Theorem 3, (11) has bounded nonoscillatory solutions. In this
case (11) may have unbounded nonoscillatory solutions; in fact, $x(t) = \log t$ is such a solution when $\beta = 1$.

(ii) Let $0 < \alpha < 1$ and $\beta = 1$. Then (12) holds, but (11) has a nonoscillatory solution $x(t) = \log t$. This example shows that the conclusion of Theorem 1 is not true if condition (1) is violated.

A similar example illustrating Theorem 2 could easily be provided.

**Bibliography**


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