WEAK SEQUENTIAL CONVERGENCE IN $L^\infty_E$
AND DUNFORD-PETTIS PROPERTY OF $L^1_E$

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ABSTRACT. For a $\sigma$-finite measure space $(X, \mathcal{A}, \mu)$ it is proved that weak sequential convergence in $L^\infty_E$ implies almost everywhere pointwise convergence, with the weak topology on the Banach space $E$. Also it is proved that if weak and norm sequential convergence coincide in $E'$, then $L^1_E$ has the Dunford-Pettis property.

In this paper, by using the theory of lifting, some properties of weakly convergent sequences in $L^\infty_E$ are obtained. Also the Dunford-Pettis (D-P) property of $L^1_E$ is obtained for some special Banach spaces $E$.

It is always assumed, unless otherwise stated that $(X, \mathcal{A}, \mu)$ is a $\sigma$-finite complete measure space. All vector spaces are taken over the field of real numbers. We shall use $N$ for the set of positive natural numbers. Notations and terminology of [7] will be used. For a Banach space $E, E'$ will denote its topological dual; by the weak topology on $E$ we mean the topology $\sigma(E, E')$. For a normed space $(E, \| \cdot \|)$ and a mapping $f: X \to E$, $\|f\|: X \to R$ is defined by $\|f\|(x) = \|f(x)\|$. We fix a lifting $\rho: L^\infty \to M^\infty$.

For a Banach space $E, L^\infty_E, L^\infty_E, L^\infty_E[E], \text{etc.}$, are taken in the sense of [7]. We start with the following result.

**Theorem 1.** If a sequence $\{f_n\}$ converges to 0 weakly in the Banach space $(L^\infty_E, N_\infty)$, $N_\infty$ being its norm, then there exists a $\mu$-negligible set $P$ such that $f_n \to 0$ pointwise on $X \setminus P$, with the $\sigma(E, E')$ topology on $E$.

**Proof.** For any $A \in \mathcal{A}$, the mapping $T_A: L^\infty_E \to L^\infty_E$, $T_A(f) = f_{\mu A}$, is continuous with the norm topology on $L^\infty_E$. This means $T_A$ is weakly continuous and so $f_n\chi_A \to 0$ weakly in $L^\infty_E$ for each $A \in \mathcal{A}$.

Fix $\varepsilon > 0$ and $X_0 \in \mathcal{A}$ with $\mu(X_0) < \infty$. By Egorov's theorem there exists $X_\varepsilon \subset X_0$ and a sequence $\{g_n\}$ of $E$-valued, $\mathcal{A}$-simple functions such that $\|f_n\chi_{X_\varepsilon} - g_n\|_\infty < 2^{-n}$, $\forall n$ and $\mu(X_0 \setminus X_\varepsilon) < \varepsilon$ (here $\| \cdot \|_\infty$ is the supremum norm). Since $f_n\chi_{X_\varepsilon} \to 0$ weakly it follows that $g_n \to 0$ weakly in $L^\infty_E$. Let $\rho: M^\infty_E[E'] \to M^\infty_E[E']$ be the corresponding lifting [7, Proposition 1, p. 77] on $M^\infty_E[E']$ and let

$$g_n = \sum_{i=1}^{p_n} h_i^{(n)} \otimes x_i^{(n)},$$

with $h_i^{(n)} \in M^\infty, x_i^{(n)} \in E, p_n \in N, \forall n$ and $\forall i, 1 < i < p_n$. It is easy to verify...
that
\[ \rho(g_n) = \sum_{i=1}^{p_n} \rho(h_i^{(n)}) \otimes x_i^{(n)} \]

[7, Theorem 4, p. 65]. Thus there exists a \( \mu \)-negligible set \( P_n \) such that \( g_n = \rho(g_n) \) on \( X \setminus P_n \). Also the space \( L_\infty^\infty[E'] \) can be canonically identified with a subspace of \( L_\infty^\infty \) [7, Theorem 4, p. 92]. Now \( \rho \) defines an isometry \( \rho: L_\infty^\infty[E'] \to M_\infty^\infty[E'] \) with \( \| \cdot \|_\infty \) (i.e., the supremum norm) topology on \( M_\infty^\infty[E'] \), and \( \{ g_n \} \subset L_\infty^\infty[E'] \). This proves \( \rho(g_n) \to 0 \) weakly in \( M_\infty^\infty[E'] \), from which it follows that \( \rho(g_n) \to 0 \) pointwise with the \( \sigma(E, E') \) topology on \( E \). So we prove that \( g_n \to 0 \) pointwise on \( X \setminus P \) with \( \sigma(E, E') \) topology on \( E \), where \( P = \bigcup_{n=1}^{\infty} P_n \) is \( \mu \)-negligible. Let \( A = \{ x \in X_0 : f_n(x) \to 0 \text{ weakly in } E \} \). Then \( A \subset (X_0 \setminus X) \cup P \) and so \( \mu^*(A) < \mu^*(X_0 \setminus X) < \varepsilon \) (here \( \mu^* \) is the outer measure defined by \( \mu^*(Q) = \inf \{ \mu(K) : K \supseteq Q, K \in \mathcal{A} \}, \forall Q \subset X \)). Since \( \varepsilon \) is arbitrary we have \( \mu^*(A) = 0 \) and so \( A \in \mathcal{A} \). Thus \( f_n(x) \to 0 \) a.e. \( [\mu] \) weakly on \( E \) for \( x \in X_n \). Since \( \mu \) is \( \sigma \)-finite it follows that \( f_n(x) \to 0 \) a.e. \( [\mu] \) weakly on \( E \) for \( x \in X \).

**Corollary 2.** Suppose weak sequential convergence and norm sequential convergence coincide in \( E \). If a sequence \( f_n \to 0 \) weakly in \( L_\infty^\infty \), then \( \| f_n \| \to 0 \) pointwise a.e. \( \mu \) (here \( \| f_n \| : X \to E, \| f_n \|(x) = \| f_n(x) \| ) \). If, in addition, \( \mu \) is finite, then \( \int \| f_n \|^p d\mu \to 0 \) for any \( p, 0 < p < \infty \).

The proof is an immediate consequence of Theorem 1 and the Lebesgue dominated convergence theorem.

**Remark 3.** (a) A Banach space has the Schur property if weakly convergent sequences are norm convergent. It easily follows from Corollary 2 that \( E \) has the Schur property if and only if for any finite measure space \( (X, \mathcal{A}, \mu) \), weakly convergent sequences in \( L_\infty^\infty \) are almost everywhere convergent.

(b) When \( E \) is scalar, Corollary 2 follows from [9, Proposition 5]. (The referee has informed me that in this case this result is also contained in the work of Grothendieck [6].)

A Banach space \( B \) is said to have the D-P property if whenever a sequence \( x_n \to 0 \) in \( (B, \sigma(B, B')) \) and \( f_n \to 0 \) in \( (B', \sigma(B', B'')) \) then \( f_n(x_n) \to 0 \) \[4\]. We shall prove that for some special kind of Banach spaces \( E \), \( L_\infty^\infty \) has the D-P property. We first prove the following lemmas.

**Lemma 4.** Let \( H \) be the vector space of all real-valued countably additive measures on a \( \sigma \)-algebra \( \mathcal{B} \) of subsets of a set \( Y \) and let \( B_\infty \) be the vector space of all bounded, real-valued, \( \mathcal{B} \)-measurable functions on \( Y \). If a uniformly bounded sequence \( f_n \to 0 \) pointwise in \( B_\infty \), then \( f_n \to 0 \) uniformly on any relatively compact subset of \( (H, \sigma(H, B_\infty)) \).

**Proof.** With pointwise order \( E = B_\infty \) is a boundedly \( \sigma \)-order complete vector lattice. Also \( H = E^\infty \), the set of all order \( \sigma \)-continuous linear functionals on \( E \). Thus if \( H_0 \) is a compact subset of \( (H, \sigma(H, B_\infty)) \) then \( H_0 \) is compact in \( (E^\infty, \sigma(E^\infty, B_\infty)) \). The result follows now from [2, Theorem 2.8, p. 187].
THEOREM 5. Let $E$ be a Banach space such that $\sigma(E', E'')$ and norm sequential convergence coincide in $E'$. Then $L^1_E$ has D-P property.

PROOF. By [7, Theorem 7, p. 94], $(L^1_E)' = L^\infty_E[E]$. The lifting $\rho$ gives a lifting $\rho: L^\infty_E[E] \to M^\infty_E[E]$ [7, Proposition 1, p. 77]. Take a sequence $g_n \to 0$ weakly in $L^1_E$ and a sequence $\delta_n \to 0$ weakly in $L^\infty_E[E]$, with $\rho(\delta_n) = \delta_n \in M^\infty_E[E]$, $\forall n$.

$\mu_n = g_n \mu$ (i.e., $\mu_n(A) = \int_A g_n \, d\mu$) is a sequence of countably additive $E$-valued measures on $\mathbb{A}$ and the variation of $\mu_n$, $|\mu_n| = \|g_n\| \mu$. It follows from the weak convergence of $\{g_n\}$ that $\sup \|\mu_n\|(X) < \infty$. Let $Q$ be the $\sigma$-complete vector lattice, with pointwise order, of real-valued, bounded, $\mathbb{A}$-measurable functions on $X$. Then $\{\{|\mu_n|\}\} \subset Q^{\infty}$ (notations of [2]). By ([3, Theorem 1, p. 305]; [1, Theorem 2, p. 290]), $\{|\mu_n|\}$ is relatively compact in $(Q^{\infty}, \sigma(Q^{\infty}, Q))$.

We take the lifting topology $\tau_\rho$ on $X$ [7, p. 59]. The mappings $\delta_n: (X, \tau_\rho) \to (E', \sigma(E', E))$ are continuous [7, Theorem 4, p. 65 and p. 84]. Since $\delta_n \to 0$ weakly in the Banach space $L^\infty_E[E]$, there exists an $\alpha > 0$ such that $\|\delta_n\| < \alpha$, $\forall n$ (here $\|\delta_n\|: X \to \mathbb{R}^+$, $\|\delta_n\|(x) = \|\delta_n(x)\|$, $\|\cdot\|$ being norm of the dual Banach space $E'$). Take an $x \in X$ and $p \in E''$. With the supremum norm $(M^\infty_E[E], \|\cdot\|)$ is a Banach space and $p_{\delta_n} \in (M^\infty_E[E], \|\cdot\|')$. Since $\rho: L^\infty_E[E] \to M^\infty_E[E]$ is an isometry, and $\delta_n \to 0$ weakly in $L^\infty_E[E]$, we get $p \circ \delta_n(x) \to 0$. Using the equivalence of weak and norm sequential convergence we get $\|\delta_n\| \to 0$ pointwise. Since the mapping $x \to \|x\|$, $(E', \sigma(E', E)) \to \mathbb{R}$, is lower semicontinuous, the mappings $\|\delta_n\|: (X, \tau_\rho) \to \mathbb{R}$ are lower semicontinuous. Since $(X, \tau_\rho)$ is extremely disconnected, $\|\delta_n\| = \|\delta_n\|$ is a bounded continuous function on $(X, \tau_\rho)$, except on a set of first category. Since $\mu$ is a category measure [8, p. 120], $\mu$ vanishes on sets of first category. This proves the $\|\delta_n\|$ are $\mathbb{A}$-measurable. By Lemma 6, $|\mu_n(\|\delta_n\|)| \to 0$ and so $\int \|g_n\| \|\delta_n\| \, d\mu \to 0$. This implies that $\int \langle g_n, \delta_n \rangle \, d\mu \to 0$ and so the result is proved.

REMARK 6. (a) From the assumption of equivalence of norm and weak sequential convergence in $E'$, it easily follows that $E$ has the Dunford-Pettis property. It is shown in [4, Corollary 17, p. 286] that if $E$ has the Dunford-Pettis property and does not contain a subspace isomorphic to $l_1(N)$, then norm and weak sequential convergence coincide in $E'$. This result is also produced in [10].

(b) Professor J. J. Uhl, Jr. has informed me that Kevin Andrews, one of his students, has independently proved Theorem 5 by an entirely different method.

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REFERENCES


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