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# WEAK SEQUENTIAL CONVERGENCE IN $L_E^\infty$ AND DUNFORD-PETTIS PROPERTY OF $L_E^1$

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ABSTRACT. For a $\sigma$-finite measure space $(X, \mathfrak{A}, \mu)$ it is proved that weak sequential convergence in $L_E^\infty$ implies almost everywhere pointwise convergence, with the weak topology on the Banach space $E$. Also it is proved that if weak and norm sequential convergence coincide in $E'$, then $L_E^1$ has the Dunford-Pettis property.

In this paper, by using the theory of lifting, some properties of weakly convergent sequences in $L_E^\infty$ are obtained. Also the Dunford-Pettis (D-P) property of $L_E^1$ is obtained for some special Banach spaces $E$.

It is always assumed, unless otherwise stated that $(X, \mathfrak{A}, \mu)$ is a $\sigma$-finite complete measure space. All vector spaces are taken over the field of real numbers. We shall use $N$ for the set of positive natural numbers. Notations and terminology of [7] will be used. For a Banach space $E$, $E'$ will denote its topological dual; by the weak topology on $E$ we mean the topology $\sigma(E, E')$. For a normed space $(E, \|\cdot\|)$ and a mapping $f$: $X \to E$, $\|f\|$: $X \to R$ is defined by $\|f\|(x) = \|f(x)\|$. We fix a lifting $\rho$: $L^\infty \to M^\infty$.

For a Banach space $E$, $L_E^1$, $L_E^\infty$, $L_{E'}^\infty[E]$, etc., are taken in the sense of [7]. We start with the following result.

THEOREM 1. *If a sequence $\{f_n\}$ converges to* 0 *weakly in the Banach space* $(L_E^\infty, N_\infty)$, *$N_\infty$ being its norm, then there exists a $\mu$-negligible set $P$ such that $f_n \to 0$ pointwise on $X \setminus P$, with the $\sigma(E, E')$ topology on $E$.*

PROOF. For any $A \in \mathfrak{A}$, the mapping $T_A$: $L_E^\infty \to L_E^\infty$, $T_A(f) = f\chi_A$, is continuous with the norm topology on $L_E^\infty$. This means $T_A$ is weakly continuous and so $f_n\chi_A \to 0$ weakly in $L_E^\infty$ for each $A \in \mathfrak{A}$.

Fix $\varepsilon > 0$ and $X_0 \in \mathfrak{A}$ with $\mu(X_0) < \infty$. By Egorov's theorem there exists $X_\varepsilon \subset X_0$ and a sequence $\{g_n\}$ of $E$-valued, $\mathfrak{A}$-simple functions such that $\|f_n\chi_{X_\varepsilon} - g_n\|_\infty < 2^{-n}$, $\forall n$ and $\mu(X_0 \setminus X_\varepsilon) < \varepsilon$ (here $\|\cdot\|_\infty$ is the supremum norm). Since $f_n\chi_{X_\varepsilon} \to 0$ weakly it follows that $g_n \to 0$ weakly in $L_E^\infty$. Let $\rho$: $M_E^\infty[E'] \to M_E^\infty[E']$ be the corresponding lifting [7, Proposition 1, p. 77] on $M_E^\infty[E']$ and let

$$g_n = \sum_{i=1}^{p_n} h_i^{(n)} \otimes x_i^{(n)},$$

with $h_i^{(n)} \in M^\infty$, $x_i^{(n)} \in E, p_n \in N$, $\forall n$ and $\forall i$, $1 \leqslant i \leqslant p_n$. It is easy to verify

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that

$$\rho(g_n) = \sum_{i=1}^{p_n} \rho(h_i^{(n)}) \otimes x_i^{(n)}$$

[7, Theorem 4, p. 65]. Thus there exists a $\mu$-negligible set $P_n$ such that $g_n = \rho(g_n)$ on $X \setminus P_n$. Also the space $L_E^\infty[E']$ can be canonically identified with a subspace of $L_E^\infty$ [7, Theorem 4, p. 92]. Now $\rho$ defines an isometry $\rho$: $L_E^\infty[E'] \to M_E^\infty[E']$ with $\|\cdot\|_\infty$ (i.e., the supremum norm) topology on $M_E^\infty[E']$, and $\{g_n\} \subset L_E^\infty[E']$. This proves $\rho(g_n) \to 0$ weakly in $M_E^\infty[E']$, from which it follows that $\rho(g_n) \to 0$ pointwise with the $\sigma(E, E')$ topology on $E$. So we prove that $g_n \to 0$ pointwise on $X \setminus P$ with $\sigma(E, E')$ topology on $E$, where $P = \bigcup_{n=1}^\infty P_n$ is $\mu$-negligible. Let $A = \{x \in X_0: f_n(x) \nrightarrow 0 \text{ weakly in } E\}$. Then $A \subset (X_0 \setminus X_\varepsilon) \cup P$ and so $\mu^*(A) \leq \mu^*(X_0 \setminus X_\varepsilon) < \varepsilon$ (here $\mu^*$ is the outer measure defined by $\mu^*(Q) = \inf\{\mu(K): K \supset Q, K \in \mathfrak{A}\}$, $\forall Q \subset X$). Since $\varepsilon$ is arbitrary we have $\mu^*(A) = 0$ and so $A \in \mathfrak{A}$. Thus $f_n(x) \to 0$ a.e. $[\mu]$ weakly on $E$ for $x \in X_0$. Since $\mu$ is $\sigma$-finite it follows that $f_n(x) \to 0$ a.e. $[\mu]$ weakly on $E$ for $x \in X$.

COROLLARY 2. *Suppose weak sequential convergence and norm sequential convergence coincide in E. If a sequence $f_n \to 0$ weakly in $L_E^\infty$, then $\|f_n\| \to 0$ pointwise a.e. $\mu$ (here $\|f_n\|$: $X \to E$, $\|f_n\|(x) = \|f_n(x)\|$). If, in addition, $\mu$ is finite, then $\int \|f_n\|^p d\mu \to 0$ for any $p$, $0 < p < \infty$.*

The proof is an immediate consequence of Theorem 1 and the Lebesgue dominated convergence theorem.

REMARK 3. (a) A Banach space has the Schur property if weakly convergent sequences are norm convergent. It easily follows from Corollary 2 that $E$ has the Schur property if and only if for any finite measure space $(X, \mathfrak{A}, \mu)$, weakly convergent sequences in $L_E^\infty$ are almost everywhere convergent.

(b) When $E$ is scalar, Corollary 2 follows from [9, Proposition 5]. (The referee has informed me that in this case this result is also contained in the work of Grothendieck [6].)

A Banach space $B$ is said to have the D-P property if whenever a sequence $x_n \to 0$ in $(B, \sigma(B, B'))$ and $f_n \to 0$ in $(B', \sigma(B', B''))$ then $f_n(x_n) \to 0$ [4]. We shall prove that for some special kind of Banach spaces $E$, $L_E^1$ has the D-P property. We first prove the following lemmas.

LEMMA 4. *Let $H$ be the vector space of all real-valued countably additive measures on a $\sigma$-algebra $\mathfrak{B}$ of subsets of a set $Y$ and let $B_\infty$ be the vector space of all bounded, real-valued, $\mathfrak{B}$-measurable functions on $Y$. If a uniformly bounded sequence $f_n \to 0$ pointwise in $B_\infty$, then $f_n \to 0$ uniformly on any relatively compact subset of $(H, \sigma(H, B_\infty))$.*

PROOF. With pointwise order $E = B_\infty$ is a boundedly $\sigma$-order complete vector lattice. Also $H = E^{\sigma c}$, the set of all order $\sigma$-continuous linear functionals on $E$. Thus if $H_0$ is a compact subset of $(H, \sigma(H, B_\infty))$ then $H_0$ is compact in $(E^{\sigma c}, \sigma(E^{\sigma c}, B_\infty))$. The result follows now from [2, Theorem 2.8, p. 187].

THEOREM 5. *Let $E$ be a Banach space such that $\sigma(E', E'')$ and norm sequential convergence coincide in $E'$. Then $L_E^1$ has D-P property.*

PROOF. By [7, Theorem 7, p. 94], $(L_E^1)' = L_{E'}^\infty[E]$. The lifting $\rho$ gives a lifting $\rho$: $L_{E'}^\infty[E] \to M_{E'}^\infty[E]$ [7, Proposition 1, p. 77]. Take a sequence $g_n \to 0$ weakly in $L_E^1$ and a sequence $\delta_n \to 0$ weakly in $L_{E'}^\infty[E]$, with $\rho(\delta_n) = \delta_n \in M_{E'}^\infty[E]$, $\forall n$.

$\mu_n = g_n\mu$ (i.e., $\mu_n(A) = \int_A g_n \, d\mu$) is a sequence of countably additive $E$-valued measures on $\mathfrak{A}$ and the variation of $\mu_n$, $|\mu_n| = \|g_n\| \mu$. It follows from the weak convergence of $\{g_n\}$ that $\sup|\mu_n|(X) < \infty$. Let $Q$ be the $\sigma$-complete vector lattice, with pointwise order, of real-valued, bounded, $\mathfrak{A}$-measurable functions on $X$. Then $\{|\mu_n|\} \subset Q^{\sigma c}$ (notations of [2]). By ([3, Theorem 1, p. 305]; [1, Theorem 2, p. 290]), $\{|\mu_n|\}$ is relatively compact in $(Q^{\sigma c}, \sigma(Q^{\sigma c}, Q))$.

We take the lifting topology $\tau_\rho$ on $X$ [7, p. 59]. The mappings $\delta_n$: $(X, \tau_\rho) \to (E', \sigma(E', E))$ are continuous [7, Theorem 4, p. 65 and p. 84]. Since $\delta_n \to 0$ weakly in the Banach space $L_{E'}^\infty[E]$, there exists an $\alpha > 0$ such that $\|\delta_n\| \leqslant \alpha$, $\forall n$ (here $\|\delta_n\|$: $X \to R^+$, $\|\delta_n\|(x) = \|\delta_n(x)\|$, $\|\cdot\|$ being norm of the dual Banach space $E'$). Take an $x \in X$ and $p \in E''$. With the supremum norm $(M_{E'}^\infty[E], \|\cdot\|)$ is a Banach space and $p\varepsilon_x \in (M_{E'}^\infty[E], \|\cdot\|)'$. Since $\rho$: $L_{E'}^\infty[E] \to M_{E'}^\infty[E]$ is an isometry, and $\delta_n \to 0$ weakly in $L_{E'}^\infty[E]$, we get $p \circ \delta_n(x) \to 0$. Using the equivalence of weak and norm sequential convergence we get $\|\delta_n\| \to 0$ pointwise. Since the mapping $x \to \|x\|$, $(E', \sigma(E', E)) \to R$, is lower semicontinuous, the mappings $\|\delta_n\|$: $(X, \tau_p) \to R$ are lower semicontinuous. Since $(X, \tau_p)$ is extremely disconnected, $\|\delta_n\|$ = a bounded continuous function on $(X, \tau_p)$, except on a set of first category. Since $\mu$ is a category measure [8, p. 120], $\mu$ vanishes on sets of first category. This proves the $\|\delta_n\|$ are $\mathfrak{A}$-measurable. By Lemma 6, $|\mu_n|(\|\delta_n\|) \to 0$ and so $\int \|g_n\| \, \|\delta_n\| \, d\mu \to 0$. This implies that $\int\langle g_n, \delta_n\rangle \, d\mu \to 0$ and so the result is proved.

REMARK 6. (a) From the assumption of equivalence of norm and weak sequential convergence in $E'$, it easily follows that $E$ has the Dunford-Pettis property. It is shown in [4, Corollary 17, p. 286] that if $E$ has the Dunford-Pettis property and does not contain a subspace isomorphic to $l_1(N)$, then norm and weak sequential convergence coincide in $E'$. This result is also produced in [10].

(b) Professor J. J. Uhl, Jr. has informed me that Kevin Andrews, one of his students, has independently proved Theorem 5 by an entirely different method.

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