A NOTE ON $M$-IDEALS IN $B(X)$

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ABSTRACT. In this paper we prove some properties of $M$-ideals and HB-subspaces in an arbitrary Banach space. We then apply these properties to prove a theorem which generalizes to other spaces Smith's and Ward's results in [8]: for $1 < p < \infty$, $B(l_p)$ contains no nontrivial summands and that each nontrivial $M$-ideal in $B(l_p)$ contains $K(l_p)$.

Introduction. A closed subspace $J$ of a Banach space $Y$ is said to be an $M$-ideal of $Y$ if its annihilator $J^\perp$ is $l_1$ complemented in $Y^*$. That is, there exists a subspace $J_*$ of $Y^*$ such that $Y^* = J^\perp \oplus J_*$ and $\|p + q\| = \|p\| + \|q\|$ whenever $p \in J^\perp$ and $q \in J_*$. $J$ is said to be an $M$-summand if $J$ is complemented by a closed subspace $J'$ such that $\|p + q\| = \max(\|p\|, \|q\|)$ whenever $p \in J$ and $q \in J'$. $M$-summands are $M$-ideals, though the reverse is not necessarily true. These concepts, first introduced for real Banach spaces in [1], also apply to complex Banach spaces. Recently, much interest has focused on the approximation properties of $M$-ideals [5], [7].

For a Banach space $X$, let $K(X)$ and $B(X)$ denote the spaces of compact operators and all bounded operators respectively. In [3], Hennefeld showed that for $X = c_0$ or $l_p$, $1 < p < \infty$, $K(X)$ is an $M$-ideal in $B(X)$. In [8], Smith and Ward proved that, for $1 < p < \infty$, $B(X)$ contains no nontrivial $M$-summands, and that any nontrivial $M$-ideal must contain $K(l_p)$. Their proof used Tam's characterization of Hermitian operators in $B(l_p)$, $p \neq 2$, the fact that $K(l_p)$ is the only two-sided ideal in $B(l_p)$, and their technique of investigating Banach algebra (with identity) $M$-ideals by looking at the associated Hermitian projections (this technique involves consideration of $B(l_p)^{**}$ and the Arens multiplication). In our proof of the generalization of the Smith-Ward result, we use instead some elementary properties of $M$-ideals and HB-subspaces, given in §1, and certain manipulations on matrices.

1. Some properties of $M$-ideals and HB-subspaces. The notion of HB-subspaces, first defined in [4], is a generalization of $M$-ideals. Moreover, in [4], it was shown that for certain Banach spaces $K(X)$ is only an HB-subspace, not an $M$-ideal, in $B(X)$.

DEFINITION 1.1. A closed subspace $H$ of a Banach space $Y$ is called an HB-subspace if its annihilator $H^\perp$ is complemented by a subspace $H_*$ such that for each $f \in Y^*$, $\|f\| > \|f_\perp\|$ and $\|f\| > \|f_*\|$ whenever $f = f_* + f_\perp$ with $f_* \in H_*$ and $f_\perp$ nonzero $\in H^\perp$.
We then have the following straightforward lemmas, some of which we merely state without proof.

**Lemma 1.2.** If $H$ is an HB-subspace of $Y$, then each $\phi \in H^*$ has a unique norm-preserving extension to $Y$.

**Lemma 1.3.** Let $H$ be an HB-subspace. Then $f \in H_\ast \Leftrightarrow \|f/H\| = \|f\|$.

**Proof.** ($\Leftarrow$) Let $f$ satisfy $\|f/H\| = \|f\|$. Write $f = f_\ast + f_\perp$. For $\varepsilon > 0$, $\exists$ norm one $x \in H$: $\|f\| - \varepsilon < f(x) = f_\ast(x) + f_\perp(x) = f_\ast(x)$. Hence, $\|f_\ast\| = \|f\|, \|f_\perp\| = 0$ and $f = f_\ast$.

($\Rightarrow$) For $f \in H_\ast$, let $g = f/H$ and $\hat{g}$ be a Hahn-Banach extension of $g$ to $Y$. By the previous part of the proof, $\hat{g} \in H_\ast$. But $f - g$ is in both $H_\ast$ and $H^\perp$, which implies $f - \hat{g} = 0$. Thus, $\|f/H\| = \|f\|$.

The proof of the above lemma shows how to obtain the decomposition for an arbitrary $g \in Y^*$, namely: $g_\ast$ is the unique Hahn-Banach extension of $g$ restricted to $H$, and $g_\perp = g - g_\ast$. Hence, we have the following lemma.

**Lemma 1.4.** If $H$ is an HB-subspace, then $H_\ast$ is isometric to $H^*$.

**Lemma 1.5.** If $H$ is an HB-subspace, and $J$ is an M-ideal with $H_\ast \subset J_\ast$, then $H \subset J$.

**Proof.** First, we claim that $J^\perp \subset H^\perp$. To see this, suppose $g \neq 0$ is in $J^\perp$. Write $g = g_{H_\ast} + g_{H^\perp}$. Note that $g_{H^\perp}$ cannot be 0, since $H^\ast \subset J_\ast$; also if $g_{H_\ast} = 0$, then we are finished. Hence, we can suppose $g_{H_\ast}$ and $g_{H^\perp}$ are both nonzero. Then,

$$\| -g_{H_\ast} + g \| = \| g_{H^\perp} \| = \| -g_{H_\ast} \| + \| g_{H^\perp} \| \quad \text{(since $g_{H_\ast}$ is nonzero)}$$

$$< \| -g_{H_\ast} \| + \| g \| \quad \text{(since $H$ is an HB-subspace)}.$$

But $-g_{H_\ast} \in J_\ast, g \in J^\perp$ contradicts the fact that $J$ is an $M$-ideal. Hence, $J^\perp \subset H^\perp$. Finally, $H \subset J$, since $H = H^\perp \cap Y, J = J^\perp \cap Y$.

**Lemma 1.6.** Let $J$ be an $M$-ideal of $Y$ and $f \in Y^*$. Then $f$ is an extreme point of the unit ball of $Y^* \Leftrightarrow f$ is in $J_\ast$ or $J^\perp$ and is an extreme point of the unit ball of $J_\ast$ or $J^\perp$.

**Lemma 1.7.** Let $H$ be an HB-subspace and $J$ an $M$-ideal. If $f \in H_\ast$ is an extreme point of the unit ball of $H^*$, then $f$ is in $J_\ast$ or $J^\perp$.

2. The generalization.

**Definition 2.1.** A basis $\{e_i\}$ is called shrinking if the biorthogonal functionals $\{e^*_i\}$ form a basis for $X^*$.

**Definition 2.2.** A basis $\{e_i\}$ for a Banach space is called unconditionally monotone if $\|\sum_{i \in A \cup B} a_i e_i\| \geq \|\sum_{i \in A} a_i e_i\|$ for all $A$ and $B$.

If $X$ has a shrinking basis $\{e_i\}$, then it follows from [6] that the operators with finite matrices are norm dense in $K(X)$. Hence, in this case, we can associate a matrix to each $f \in K(X)^*$ such that $f$ is determined by its matrix.
Lemma 2.3. Let $X$ have an unconditionally monotone, shrinking basis.

1. For each $f \in K(X)^*$, the functional obtained from the matrix of $f$ by replacing with zeros any set of rows or columns will have norm $< \|f\|$. 

2. If a matrix in $K(X)$ consists of a single nonzero column (row), its norm in $K(X)$ is equal to its norm as an element of $X$ ($X^*$).

3. If a matrix in $K(X)^*$ consists of a single nonzero column (row), its norm in $K(X)^*$ is equal to its norm as an element of $X^*$ ($X^{**}$).

These facts are proved in [2].

Definition 2.4. We shall call a basis $\{e_i\}$ uniformly smooth if, for each $\varepsilon > 0$, $\exists \delta > 0$ such that $\|x + y\| < 1 + \varepsilon\|y\|$ whenever $x$ and $y$ have disjoint supports, $\|x\| = 1$ and $\|y\| < \delta$. We shall call $\{e_i\}$ quasi-uniformly smooth if, for each $\varepsilon > 0$, $\exists \delta > 0$ such that $\|e_i + \lambda e_j\| < 1 + \delta\varepsilon$ for all $i, j$, whenever $|\lambda| < \delta$. Note that if a basis is uniformly smooth, the Banach space itself need not be uniformly smooth. For example, consider the standard basis for $c_0$.

The following is a generalization of the Smith-Ward result, since the hypotheses of the theorem are satisfied if $X$ is $l_p$, $1 < p < \infty$.

Theorem 2.5. Let $X$ be a Banach space with an unconditionally monotone, uniformly smooth basis $\{e_i\}$ and with $\{e_i^*\}$ a quasi-uniformly smooth basis for $X^*$. Then any nontrivial $M$-ideal in $B(X)$ must contain $K(X)$, and $B(X)$ does not contain any nontrivial $M$-summands.

Proof. Let $f_{ij}$ denote the functional with a one in the $ij$ place and zeros elsewhere. We claim that $[f_{ij}; \text{all } ij] = K(X)^*$. For suppose the contrary, i.e., suppose that there exists an $f \in K(X)^*$ which is not a uniform limit of finite matrix elements of $K(X)^*$. Since $\{e_i\}$ is shrinking, we can assume w.l.o.g. that $\|f_n\| < 1$, where $f_n$ is the functional formed from $f$ by deleting the first $n$ rows and columns from the matrix for $f$. Pick $\delta < 1$ corresponding to $\varepsilon = 1/2$ in the definition of a uniformly smooth basis. Then pick $N$ such that $\|f_N\| < (1 + \frac{3}{4}\delta)/(1 + \frac{1}{2}\delta)$ and choose $T$ and $U$ norm one, disjoint operators (i.e., $\exists m$ such that $t_{ij} = 0$ if $i$ or $j > m$ and $u_{ij} = 0$ if $i$ or $j < m$) with both $f_N(T)$ and $f_N(U) > 1 - \delta/8$. Then,

$$\frac{f_N(T + \delta U)}{\|T + \delta U\|} > \frac{1 + \frac{3}{4}\delta}{1 + \frac{1}{2}\delta},$$

which is a contradiction. Hence, $[f_{ij}; \text{all } ij] = K(X)^*$.

Each $f_{ij}$ must be extreme in the unit ball of $K(X)^*$, for suppose that $f_{ij} + g$ has a one in the $ij$ place and an $\varepsilon > 0$ in the $kl$ place. For this $\varepsilon$, let $\delta$ be the smaller of the smoothness $\delta$'s for $\{e_i\}$ and $\{e_i^*\}$. Then for $T$, the operator with $t_{ij} = 1$, $t_{kl} = \delta$, and zeros elsewhere, we have $(f_{ij} + g)T = 1 + \delta\varepsilon$ and $\|T\| < 1 + \delta\varepsilon$.

In [4] it was shown that if $X$ has an unconditionally monotone, uniformly smooth basis, then $K(X)$ is an HB-subspace of $B(X)$.

Now suppose that $J$ is a nontrivial $M$-ideal in $B(X)$. Each $f_{kl}$ is extreme in the unit ball of $K(X)^*$ and hence, by Lemma 1.7, each $f_{kl}$ must be in $J_*$ or
$J^\perp$. Let $T \neq 0$ be in $J$ and pick $f_{ij}$: $f_{ij}(T) \neq 0$. Then $f_{ij}$ must be in $J_*$. Next suppose that some $f_{mn} \in J^\perp$. This would contradict the fact that $J$ is an $M$-ideal, since $\|f_{ij} + f_{mj}\|$ and $\|f_{mn} + f_{mj}\|$ both have norm less than 2 by Lemma 2.3 and the smoothness hypotheses. Thus, $[f_{ij}: \text{all } i, j] \subset J_*$ and by Lemma 1.5 $K(X) \subset J$.

$B(X)$ has no nontrivial $M$-summands since, for each norm one $U \in B(X)$, $\exists$ an operator $E_{ij}$ with a one in the $ij$ place and zeros elsewhere such that $\|U + E_{ij}\| > 1$.

**Corollary 2.6.** For $X = d(a, p)$, any Lorentz sequence space with $1 < p < \infty$, the hypotheses of Theorem 2.5 are satisfied.

**Proof.** To see that the basis $\{e_i^*\}$ is quasi-uniformly smooth, note that for each $\delta > 0$, $e_i^* + \delta e_j^*$ will achieve its norm on an element of the form $(e_i + \lambda_\delta e_j)/\|e_i + \lambda_\delta e_j\|$, such that $\lambda_\delta \to 0$ as $\delta \to 0$. The basis $\{e_i\}$ is uniformly smooth, since $\|x + y\|_p < \|x\|_p + \|y\|_p$, whenever $x$ and $y$ are disjoint.

**Corollary 2.7.** For each $j$ let $X_j$ be a space with an unconditionally monotone, uniformly smooth basis $\{e_i^j\}$ and a quasi-uniformly smooth basis $\{e_i^{*j}\}$ such that for each $e > 0$, there is a common smoothness $\delta$ for all $j$. Then the hypotheses of Theorem 2.5 are satisfied for $(\sum_{i=1}^{\infty} \oplus X_j)_p$.

**Bibliography**


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