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GAUSSIAN MEASURE OF LARGE BALLS IN A HILBERT SPACE[1]

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ABSTRACT. Let P be a zero mean Gaussian measure in a Hilbert space. The asymptotic behavior of $P\{\|x - b\|^2 > \varepsilon\}$ as $\varepsilon \to \infty$ is studied in this note.

Let P be a Gaussian measure in a separable Hilbert space $\mathcal{H}$ and b be an arbitrary fixed element in $\mathcal{H}$. We shall study how fast $P\{\|x - b\|^2 > \varepsilon\} \to 0$ as $\varepsilon \to \infty$.

Without loss of generality, P is assumed to be of mean 0. Let B be the covariance operator of P, and let the eigenvalues (corresponding eigenvectors) of B be ordered by $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ ($\{e_i\}$). Let k be the multiplicity of the largest eigenvalue, $x_i = \langle x, e_i \rangle$, $b_i = \langle b, e_i \rangle$ and $a = (\sum_1^k b_i^2)^{1/2}$. Finally, let F denote the distribution function of $\|x - b\|^2 = \sum_1^\infty (x_i - b_i)^2$. The Laplace transform of F is

$$\phi(c) = \int_0^\infty c^{-ct}\, dF(t) = \prod_{i=1}^{\infty} \left[(1 + 2c\lambda_i)^{-1/2} \exp\left(-b_i^2 c(1 + 2c\lambda_i)^{-1}\right)\right],$$

$$\begin{aligned}\psi(c) &= \int_0^\infty e^{-ct + t/2\lambda_1}(1 - F(t))\, dt \\ &= \left(1 - \phi\left(c - \frac{1}{2\lambda_1}\right)\right)\left(c - \frac{1}{2\lambda_1}\right)^{-1} \\ &= \left(c - \frac{1}{2\lambda_1}\right)^{-1} (2c\lambda_1)^{-k/2} \exp\left(\frac{a^2}{4c\lambda_1^2} - \frac{a^2}{2\lambda_1}\right) \\ &\quad \times \left[(2c\lambda_1)^{k/2} \exp\left(\frac{a^2}{2\lambda_1} - \frac{a^2}{4c\lambda_1^2}\right) - \prod_{k+1}^{\infty} \left(1 + 2c\lambda_i - \frac{\lambda_i}{\lambda_1}\right)^{-1/2}\right. \\ &\qquad \left. \times \exp - \sum_{k+1}^{\infty} b_i^2 \left(c - \frac{1}{2\lambda_1}\right)\left(1 + 2c\lambda_i - \frac{\lambda_i}{\lambda_1}\right)^{-1}\right]. \quad (1)\end{aligned}$$

If $a = 0$, then

$$\psi(c) \sim (2\lambda_1)^{1-k/2} \prod_{k+1}^{\infty} \left(1 - \frac{\lambda_i}{\lambda_1}\right)^{-1/2} \exp\left(\frac{1}{2\lambda_1} \sum_{k+1}^{\infty} b_i^2 \left(1 - \frac{\lambda_i}{\lambda_1}\right)\right)^{-1} c^{-k/2}$$

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as $c \to 0$. By a Tauberian theorem [1], as $\varepsilon \to \infty$,

$$\int_0^\varepsilon e^{t/2\lambda_1}(1 - F(t))\,dt \sim \frac{1}{\Gamma(k/2+1)}(2\lambda_1)^{1-k/2} \prod_{k+1}^{\infty}\left(1 - \frac{\lambda_i}{\lambda_1}\right)^{-1/2} \times \exp\left(\frac{1}{2\lambda_1}\sum_{k+1}^{\infty} b_i^2\left(1 - \frac{\lambda_i}{\lambda_1}\right)^{-1}\right)\varepsilon^{k/2}.$$

Since $e^{t/2\lambda_1}(1 - F(t))$ is continuous and positive, L'Hôpital's rule is applicable. Hence we have

THEOREM 1. *If* $a = (\Sigma_1^k b_i^2)^{1/2} = 0$, *then*

$$P\left\{\|x - b\|^2 > \varepsilon\right\} \sim K_1 e^{-\varepsilon/2\lambda_1}\varepsilon^{k/2-1},$$

where

$$K_1 = \frac{1}{\Gamma(k/2)}(2\lambda_1)^{1-k/2} \prod_{k+1}^{\infty}\left(1 - \frac{\lambda_i}{\lambda_1}\right)^{-1/2} \exp \frac{1}{2\lambda_1}\sum_{k+1}^{\infty} b_i^2\left(1 - \frac{\lambda_i}{\lambda_1}\right)^{-1}. \tag{2}$$

If $a > 0$, then $\psi(c) \to \infty$ exponentially fast as $c \to 0$. The ordinary Tauberian theorem is not applicable here. We shall try another approach.

Let $\bar{b} \in \mathcal{H}$ such that $\bar{b}_i = 0$ if $i \leqslant k$ and $\bar{b}_i = b_i$ otherwise. Then, for ε large enough

$$P\left\{\|x - b\|^2 > \varepsilon\right\} \leqslant P\left\{\|x - \bar{b}\|^2 > (\sqrt{\varepsilon} - a)^2\right\} \sim K_1\left(\exp \frac{-1}{2\lambda_1}(\sqrt{\varepsilon} - a)^2\right)\varepsilon^{k/2-1}.$$

On the other hand, $P\{\|x - b\|^2 > \varepsilon\}$ may be regarded as $P_b\{\|x\|^2 > \varepsilon\}$, where P_b has mean $-b$ and covariance B. It is easily seen that P_b is equivalent to $P_{\bar{b}}$ and

$$\frac{dP_b}{dP_{\bar{b}}}(x) = \exp\left(\frac{-\Sigma_1^k x_i b_i}{\lambda_1} - \frac{a^2}{2\lambda_1}\right).$$

Therefore

$$\begin{aligned}
P\left\{\|x - b\|^2 > \varepsilon\right\} &= P_b\left\{\|x\|^2 > \varepsilon\right\} \\
&= \int_{\|x\|^2 > \varepsilon} \exp\left(\frac{-\Sigma_1^k x_i b_i}{\lambda_1} - \frac{a^2}{2\lambda_1}\right) dP_{\bar{b}}(x) \\
&\geqslant (2\pi\lambda_1)^{-k/2}\int_{x_1^2 + \cdots + x_k^2 > \varepsilon} \exp\left(\frac{-1}{2\lambda_1}\sum_{i=1}^k (x_i + b_i)^2\right) dx_1 \cdots dx_k \\
&= (2\pi\lambda_1)^{-k/2}\int_{x_1^2 + \cdots + x_k^2 > \varepsilon} \\
&\qquad \times \exp\left(\frac{-1}{2\lambda_1}\left[(x_1 - a)^2 + x_2^2 + \cdots + x_k^2\right]\right) dx_1 \cdots dx_k.
\end{aligned} \tag{3}$$

(To get the last equality, just rotate the vector $(-b_1, \ldots, -b_k)$ to the first coordinate.)

For $k \geq 3$, change the last $k-1$ coordinates to polar coordinates; (3) becomes

$$(2\pi\lambda_1)^{-k/2}\frac{2\pi^{(k-1)/2}}{\Gamma((k-1)/2)}\int_{\substack{x_1^2+r^2>\varepsilon\\ r>0}}\left(\exp-\frac{(x_1-a)^2+r^2}{2\lambda_1}\right)r^{k-2}\,dr\,dx_1$$

$$= (2\pi\lambda_1)^{-k/2}\frac{2\pi^{(k-1)/2}}{\Gamma((k-1)/2)}$$

$$\times\int_0^{\pi}\int_{\sqrt{\varepsilon}}^{\infty}\left(\exp\frac{\rho^2+a^2-2\rho a\cos\theta}{2\lambda_1}\right)\rho(\rho\sin\theta)^{k-2}\,d\rho\,d\theta.$$

By L'Hôpital's rule

$$\lim_{\varepsilon\to\infty}\frac{\int_0^{\pi}\int_{\sqrt{\varepsilon}}^{\infty}(\exp((2\rho a\cos\theta-\rho^2-a^2)/2\lambda_1))\rho(\rho\sin\theta)^{k-2}\,d\rho\,d\theta}{\varepsilon^{(k-3)/4}\exp\left(-(\sqrt{\varepsilon}-a)^2/2\lambda_1\right)}$$

$$= \lim_{\varepsilon\to\infty}\frac{\varepsilon^{(k-2)/2}(\exp-(\varepsilon+a^2)/2\lambda_1)\int_0^{\pi}(\exp(\sqrt{\varepsilon}\,a\cos\theta)/\lambda_1)(\sin\theta)^{k-2}\,d\theta}{\lambda_1^{-1}\varepsilon^{(k-3)/4}\exp\left(-(\sqrt{\varepsilon}-a)^2/2\lambda_1\right)}. \tag{4}$$

And, by Laplace's method [2]

$$\int_0^{\pi}\exp\frac{\sqrt{\varepsilon}\,a\cos\theta}{\lambda_1}(\sin\theta)^{k-2}\,d\theta\sim\frac{1}{2}\Gamma\left(\frac{k-1}{2}\right)\left(\frac{2\lambda_1}{a\sqrt{\varepsilon}}\right)^{(k-1)/2}\exp\frac{\sqrt{\varepsilon}\,a}{\lambda_1}.$$

Hence (4) is

$$\frac{1}{2}\lambda_1\Gamma\left(\frac{k-1}{2}\right)\left(\frac{2\lambda_1}{a}\right)^{(k-1)/2},$$

and, consequently, (3) is asymptotic to

$$\sqrt{\frac{\lambda_1}{2\pi}}\,a^{(1-k)/2}\varepsilon^{(k-3)/4}\exp-\frac{1}{2\lambda_1}(\sqrt{\varepsilon}-a)^2.$$

(The special cases $k = 1$ and $k = 2$ easily lead to the same expression.)

To sum up, we have

THEOREM 2. *If $a > 0$, then*

$$\limsup_{\varepsilon\to\infty} P\left\{\|x-b\|^2>\varepsilon\right\}\varepsilon^{1-k/2}\exp\frac{1}{2\lambda_1}(\sqrt{\varepsilon}-a)^2 \leq K_1,$$

where K_1 is defined by (2);

$$\liminf_{\varepsilon\to\infty} P\left\{\|x-b\|^2>\varepsilon\right\}\varepsilon^{(3-k)/4}\exp\frac{1}{2\lambda_1}(\sqrt{\varepsilon}-a)^2 \geq \sqrt{\frac{\lambda_1}{2\pi}}\,a^{(1-k)/2}.$$

REMARKS. 1. Zolotarev discussed the limiting behavior of $P\{\sum_1^\infty x_i^2 > \varepsilon\}$ as $\varepsilon \to \infty$ in [**4**], where $\{x_i\}$ are independent $\mathfrak{N}(0, \sigma_i^2)$ with $\sum_1^\infty \sigma_i^2 < \infty$. This is a special case of Theorem 1. The proof here is much simpler.

2. It is tempting to try to apply Theorem 3 of [**4**], but this would require establishing that $e^{\varepsilon/(2\lambda_1)}(1 - F(\varepsilon))$ is nondecreasing, and would still give a weakened version of Theorem 2. Of course, our theorem also leaves open the question of the exact asymptotic behavior of $1 - F(\varepsilon)$, when $a > 0$.

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