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COMPACTIFICATIONS WITH COUNTABLE REMAINDER

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ABSTRACT. In this paper, we deal with the problem of characterizing those spaces that have a compactification with countable remainder.

1. Introduction and definitions. A collection $\mathcal{Q}$ of subsets of a topological space X is called a network if every open subset of X is the union of a subcollection of $\mathcal{Q}$. $R(X)$ denotes the set of all points of X which possess no compact neighbourhood. If Y is a Hausdorff compactification of X, it is readily seen that $R(X)$ is the intersection of X with the closure of $Y - X$ in Y. A Hausdorff compactification Y of X is said to have *countable remainder* if $Y - X$ is a countable set; by an abuse of terminology, we shall say that such a Y is a *countable compactification* of X. In what follows, the space X is assumed to be at least Tychonoff. Two necessary conditions for X to have a countable compactification are (a) X is Čech-complete and (b) X is rim-compact. These are, in fact, sufficient conditions as well in the case when X is metric separable **[6]**, **[10]**. However, the product of the space of irrational numbers with an uncountable discrete space, despite satisfying both (a) and (b), possesses no countable compactification **[4]**. There has recently been interest in finding conditions which, together with (a) and (b), ensure that X has a countable compactification (**[2]**, **[3]**, **[4]**, **[8]**). Terada has shown that one such condition is that $R(X)$ is compact metric, and Hoshina has weakened this to the requirement that $R(X)$ is metric separable. In this paper, we show that (a) and (b), together with the condition that $R(X)$ has a countable network, ensure that X has a countable compactificaion. This includes Hoshina's result as well as the case when $R(X)$ is countable. In addition, our proof is considerably shorter than the one given by Hoshina. Furthermore, we construct examples to show that, in general, the topological properties of $R(X)$ do not determine whether X has a countable compactification.

2. A result.

THEOREM. *Let X be a Čech-complete, rim-compact space such that $R(X)$ has a countable network. Then X has a countable compactification.*

PROOF. Since X is rim-compact, X has at least one compactification Z with $\operatorname{ind}(Z - X) \leq 0$, where ind denotes small inductive dimension, and since X

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is Čech-complete, $Z - X = \bigcup_{n=1}^{\infty} F_n$, where for each n in N, the set of positive integers, F_n is compact [5]. Let $\{A_n: n \in N\}$ be a network for $R(X)$. For a fixed n in N, let $M = \{m \in N: \bar{A}_m \cap F_n = \emptyset\}$. If x is a point of $R(X)$, by regularity of Z, there is an open set V of Z and some m in M with $x \in A_m \subset V \subset \bar{V} \subset Z - F_n$. For each m in M, by normality of Z, there is a cozero set G_m of Z with $A_m \subset G_m \subset Z - F_n$. Put

$$E_n = Z - \bigcup_{m \in M} G_m \cup (X - R(X)).$$

It is readily seen that E_n is a compact subset of $Z - X$ such that $F_n \subset E_n$, $Z - X = \bigcup_{n=1}^{\infty} E_n$ and the complement of E_n in any compact subset of $\overline{Z - X} = (Z - X) \cup R(X)$ is σ-compact. We may further assume that $E_n \subset E_{n+1}$ for each n in N. Now $E_{n+1} - E_n$ is a locally compact, σ-compact space with $\operatorname{ind}(E_{n+1} - E_n) \leq 0$. Hence $E_{n+1} - E_n$ is the union of a countable collection of mutually disjoint compact sets. It follows that $Z - X = \bigcup_{n=1}^{\infty} B_n$, where, for n, m in N with $n \neq m$, B_n, B_m are disjoint compact sets, and $(Z - X \cup B_n) \cup R(X) = \bigcup_{m=1}^{\infty} C_{n,m}$, where $C_{n,m}$ is compact for all n, m in N.

Since $Z - X$ is Lindelöf and $\operatorname{ind}(Z - X) \leq 0$, then $\dim(Z - X) \leq 0$, where dim denotes covering dimension. Hence, if E, F are disjoint closed sets of Z, there exist disjoint open sets G, H with $E \subset G$, $F \subset H$ and $Z - X \subset G \cup H$ (see e.g. [1, Proposition 4]). It follows that there are pairs G_i, H_i of disjoint open sets of Z with $(Z - X) \subset G_i \cup H_i$, $i \in N$, and such that $E \subset G_i$ and $F \subset H_i$ for some i in N in each of the following cases. Firstly when $E = B_n$ and $F = C_{n,m}$, secondly when $E = \bar{A}_n$, $F = \bar{A}_m$ and $\bar{A}_n \cap \bar{A}_m = \emptyset$, and thirdly when $E = \bar{A}_n$, $F = B_m$ and $\bar{A}_n \cap B_m = \emptyset$, where n, m are in N.

We now define an equivalence relation $\sim$ on Z as follows. If $x, y \in B_n$ for some n in N, then $x \sim y$ if and only if x and y belong to the same member of $\{G_i, H_i\}$ for each $i \leq n$. Otherwise, $x \sim y$ if and only if $x = y$. Let $\pi: Z \to Y$ be the quotient map induced by $\sim$. The equivalence class $\pi^{-1}\pi(x)$ of a point x of B_n is the closed set $D_1 \cap \cdots \cap D_n \cap B_n$, where, for $i \leq n$, D_i is the member of $\{G_i, H_i\}$ which contains x. Hence $\pi(B_n)$ consists of a finite number of points. Clearly, Y is a T_1 compactification of X with $Y - X$ countable. To complete the proof, it suffices to show that π is a closed map, since this implies that Y is normal and therefore Hausdorff.

Let S be a closed set of Z. Then $\pi^{-1}\pi(S) = S \cup T$, where $T = \bigcup_{n=1}^{\infty} T_n$ and $T_n = \pi^{-1}\pi(S \cap B_n) - S$. Let x be a limit point of T. It suffices to show that $x \in S \cup T$, since this implies that $\pi^{-1}\pi(S)$ is closed and hence π is closed. Since T is a subset of the closed set $(Z - X) \cup R(X)$, either $x \in R(X)$ or, for some n in N, $x \in B_n$. We note that, for m, k in N, since $\pi(B_m)$ is finite, then $\pi^{-1}\pi(S \cap B_m)$ is closed, so that if x is not in $\bigcup_{m<k} \pi^{-1}\pi(S \cap B_m)$, then x is a limit point of $\bigcup_{m>k} T_m$.

We first assume that $x \in R(X)$. Let $K = \{k \in N: x \in G_k \cup H_k\}$. For k in

K, write D_k for the element of $\{G_k, H_k\}$ which contains x. Now x is a limit point of $\bigcup_{m>k} T_m$ and hence there is an element x_k of this set which is contained in $\bigcap (D_i\colon i \in K, i \leq k)$. Let y_k be an element of S with $y_k \sim x_k$. Then, for $i \leq k$, $y_k \in H_i$ implies $x_k \in H_i$. The infinite subset $\{y_1, y_2, \ldots\}$ of the compact set S has a limit point y in S. Suppose $y \neq x$. Either $y \in R(X)$ or $y \in B_n$ for some n in N. In the first case, there are open neighbourhoods U, V of x, y with $\bar{U} \cap \bar{V} = \varnothing$ and m, n in N with $x \in A_m \subset U$ and $y \in A_n \subset V$. Clearly $\bar{A}_m \cap \bar{A}_n = \varnothing$ and hence there is r in N with $\bar{A}_m \subset G_r$ and $\bar{A}_n \subset H_r$. In the second case, let U be a neighbourhood of x with $\bar{U} \cap B_n = \varnothing$ and let m be in N with $x \in A_m \subset U$. Since $\bar{A}_m \cap B_n = \varnothing$, there is an r in N with $\bar{A}_m \subset G_r$ and $B_n \subset H_r$. Now since y is a limit point of $\{y_1, y_2, \ldots\}$, for some $k > r$, $y_k \in H_r$, which implies that $x_k \in H_r$, so that, since $G_r \cap H_r = \varnothing$, $x_k \notin G_r = D_r$. This contradicts the fact that x_k is in $\bigcap(D_i\colon i \in K, i \leq k)$ and shows that $x = y$ and hence $x \in S$.

Finally, suppose $x \in B_n$ for some $n \in N$. It remains to show that $x \in \pi^{-1}\pi(S \cap B_n)$. Suppose this is false. For $i \in N$, let D_i be the member of $\{G_i, H_i\}$ which contains x. Then $\pi^{-1}\pi(x) = D_1 \cap \ldots \cap D_n \cap B_n$ and $S \cap D_1 \cap \ldots \cap D_n \cap B_n = \varnothing$. The closure Q of $(S - X) \cap D_1 \cap \ldots \cap D_n$ is a compact subset of $(Z - X) \cup R(X)$ which is disjoint from B_n. For if $y \in B_n \cap Q$, then $y \in B_n \cap S$, so that for some $j \leq n$, $y \notin D_j$, and if P_j is the member of $\{G_j, H_j\}$ which contains y, then $P_j \cap Q = \varnothing$. Thus Q is a compact subspace of $\bigcup_{k=1}^{\infty} C_{n,k}$. Hence there is a finite subset L of N such that $B_n \subset G_i$ for each $i \in L$ and $Q \subset \bigcup (H_i\colon i \in L)$. Let $k = n + \max L$ and $D = D_1 \cap \ldots \cap D_k$. Since $x \in B_n$, for $i \in L$, $D_i = G_i$. Let $m > k$ and suppose $y \in D \cap T_m$. Then there is z in $S \cap B_m$ with $y \sim z$. For $i \leq k$, y and z belong to the same element of $\{G_i, H_i\}$. Hence $z \in D$ and it follows that $z \in Q$. Therefore for some i in L, $z \in H_i$, which is absurd since $G_i \cap H_i = \varnothing$ and $z \in D \subset D_i = G_i$. This shows that x is not a limit point of $\bigcup_{m>k} T_m$ and since our assumption that $x \in B_n$ and $x \notin \pi^{-1}\pi(S \cap B_n)$ implies that x is not in $\bigcup_{m\leq k} \pi^{-1}\pi(S \cap B_m)$, then x is not a limit point of T. This contradiction shows that x must be in $\pi^{-1}\pi(S \cap B_n)$ and completes the proof of the theorem.

3. Some examples. Example 1 shows that there are rim-compact, Čech-complete spaces X, X_1, such that, despite $R(X)$, $R(X_1)$ being homeomorphic, X has a countable compactification but not X_1. In this example, $R(X)$ is compact. In Example 2, the same pathology is exhibited with $R(X)$ discrete. Hoshina **[4]** has shown that if a paracompact space X has a countable compactification, then $R(X)$ is Lindelöf. Example 2 shows that, in general, the fact that X has a countable compactification does not imply that $R(X)$ is Lindelöf.

We need the following result of Hoshina **[4]**.

LEMMA. *If X has a countable compactification and $\mathfrak{U}$ is a collection of mutually disjoint open sets of X with $U \cap R(X) \neq \varnothing$ for each U in $\mathfrak{U}$, then $\mathfrak{U}$ is countable.*

EXAMPLE 1. Let R be the set of real numbers with the usual topology. Then $X = \beta R - N$, where β denotes Stone-Čech compactification, has a countable compactification and $R(X) = \beta N - N$ [**8**, Example 3].

Let $N \cup \{\infty\}$ be the one-point compactification of N, $Y = (N \cup \{\infty\}) \times (N \cup \{\infty\}) \times R(X)$ and $X_1 = Y - \{\infty\} \times N \times R(X)$. Since Y is compact and $Y - X_1$ is σ-compact and zero-dimensional, then X_1 is Čech-complete and rim-compact. In addition, $R(X_1) = \{\infty\} \times \{\infty\} \times R(X)$ is homeomorphic with $R(X)$. Let $\mathcal{U}$ be an uncountable collection of mutually disjoint nonempty open sets of $\beta N - N$ [**9**, p. 77]. For each U in $\mathcal{U}$, let $U^* = (N \cup \{\infty\}) \times (N \cup \{\infty\}) \times U$. Then $\{U^* \cap X_1: U \in \mathcal{U}\}$ is an uncountable collection of mutually disjoint open sets of X_1 with $U^* \cap X_1 \cap R(X_1) \neq \varnothing$ for each U in $\mathcal{U}$. The lemma implies that X_1 has no countable compactification.

EXAMPLE 2. Let P be the set of irrational numbers and Q the set of rational numbers. For each x in P, let $\{x_1, x_2, \ldots\}$ be a sequence of rationals converging to x in the usual topology of R. A subset A of R is defined to be open if whenever $x \in A \cap P$, then there is n in N with $\{x_n, x_{n+1}, \ldots\} \subset A$. With this topology, R is locally compact and Hausdorff, Q is dense in R and P is a closed subspace of R with discrete topology [**7**, p. 87]. Let $R \cup \{\infty\}$ be the one-point compactification of R, $Y = (N \cup \{\infty\}) \times (R \cup \{\infty\})$ and $X = Y - \{\infty\} \times Q \cup \{\infty\}$. Then Y is a countable compactification of X, while $R(X) = \{\infty\} \times P$ is not Lindelöf.

Let $Z = (N \cup \{\infty\}) \times Y$ and $X_1 = (Z - \{\infty\} \times Y) \cup \{\infty\} \times \{\infty\} \times P$. Then X_1 is Čech-complete and rim-compact, because $Z - X_1$ is σ-compact and zero-dimensional, and $R(X_1) = \{\infty\} \times \{\infty\} \times P$ is homeomorphic with $R(X)$. However, the lemma implies that the closed subspace $N \times (N \cup \{\infty\}) \times (P \cup \{\infty\}) \cup R(X_1)$ of X_1 has no countable compactification, and hence X_1 has no countable compactification.

We can obviously choose X, X_1 so that $R(X)$, $R(X_1)$ are homeomorphic with the one-point compactification of P.

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