PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, February 1980

THE SMITH INVARIANTS OF A MATRIX SUM

ROBERT C. THOMPSON[1]

ABSTRACT. A divisibility relation is proved relating the invariant factors of a sum of integral matrices.

Let R be a principal ideal domain, that is, a commutative ring without zero divisors in which every ideal is principal. Let A be a matrix over R. It is well known [1] that for the relation "equivalence of matrices," namely $A \to UAV$ where U, V are invertible matrices over R, there exists a complete set of invariants, the *invariant factors* of A. If A is $n \times n$, its invariant factors are elements $h_1(A), \dots, h_n(A)$ of R satisfying the divisibility conditions $h_1(A)| \cdots |h_n(A)$. Very much, in fact, almost full information is now available [2] about the restrictions imposed upon the invariant factors of matrices A, B, C when it is required that $C = AB$. Nothing, however, seems to be known about the behavior of invariant factors when matrices are added, and the objective of this paper is to obtain a result along these lines.

THEOREM 1. *Let A, B, $C = A + B$, be matrices over R, with invariant factors*

$$h_1(A)| \cdots |h_n(A), \qquad h_1(B)| \cdots |h_n(B), \qquad h_1(C)| \cdots |h_n(C),$$

respectively. Let the invariant factors of

$$D = \operatorname{diag}(h_1(A), \dots, h_n(A), h_1(B), \dots, h_n(B))$$

be $h_1(D)| \cdots |h_{2n}(D)$. Then

$$h_i(D)|h_i(C), \qquad i = 1, \dots, n.$$

PROOF. Let $D_A = \operatorname{diag}(h_1(A), \dots, h_n(A))$. Unimodular matrices U_1, U_2, V_1, V_2 over R exist such that

$$A = U_1 D_A V_1, \qquad B = U_2 D_B V_2.$$

Then the matrix

$$M = \begin{bmatrix} U_1 & U_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_A & 0 \\ 0 & D_B \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ V_2 & I \end{bmatrix} = \begin{bmatrix} C & \cdot \\ \cdot & \cdot \end{bmatrix}$$

is $2n \times 2n$, has the same invariant factors as D and has C as an $n \times n$ submatrix. Now, a recent result of the author [3] asserts that invariant factors $h_1(M)|h_2(M)| \cdots$ of a matrix M and the invariant factors of a submatrix C

Received by the editors November 28, 1977.
AMS (MOS) subject classifications (1970). Primary 15A36.
[1]The preparation of this paper was supported in part by Grant 77-3316, United States Air Force.

satisfy

$$h_i(M)|h_i(C), \qquad i = 1, 2, \ldots .$$

Since here $h_i(M) = h_i(D)$, the proof is complete.

In fact, further information about invariant factors of submatrices is given in [3]. However, this additional information yields nothing in the present case, although it will yield additional information about the invariant factors of A, B, $A + B$, when A and B are rectangular matrices.

As an application of Theorem 1, we prove an explicit inequality relating the invariant factors of A, B, $A + B$.

THEOREM 2. *Let the invariant factors of A, B, $C = A + B$, be as defined in the statement of Theorem* 1. *Then*

$$\gcd\big(h_i(A), h_j(B)\big)|h_{i+j-1}(A + B)$$

for any indices i,j with $1 \leqslant i, j \leqslant n$, $i + j - 1 \leqslant n$, where gcd *denotes greatest common divisor.*

PROOF. It suffices to prove that $\gcd(h_i(A), h_j(B))|h_{i+j-1}(D)$, and for this we may focus attention upon elementary divisors for one prime p, i.e., upon one prime factor p of $h_n(A)h_n(B)$. Let

$$p^{e_i}\|h_i(A), p^{f_i}\|h_i(B), p^{g_i}\|h_i(D)$$

be the exact powers of p appearing in $h_i(A)$, $h_i(B)$, $h_i(D)$, respectively; $e_1 \leqslant \cdots \leqslant e_n$, $f_1 \leqslant \cdots \leqslant f_n$, $g_1 \leqslant \cdots \leqslant g_{2n}$. The power on p in the tth determinantal divisor of D is then

$$g_1 + \cdots + g_t = \min_{\substack{r,s \\ r+s=t}} (e_1 + \cdots + e_r + f_1 + \cdots + f_s). \tag{1}$$

We wish to prove that $\min(e_i, f_j) \leqslant g_{i+j-1}$. From (1) we certainly have

$$g_1 + \cdots + g_{i+j-1} = e_1 + \cdots + e_r + f_1 + \cdots + f_s \tag{2}$$

for certain r,s with $r + s = i + j - 1$. We cannot have both $r < i$ and $s < j$, so suppose $s \geqslant j$. Using (1) again, we have

$$g_1 + \cdots + g_{i+j-2} \leqslant e_1 + \cdots + e_r + f_1 + \cdots + f_{s-1}. \tag{3}$$

Combining (2) and (3), we obtain $g_{i+j-1} \geqslant f_s \geqslant f_j \geqslant \min(e_i, f_j)$. The proof is complete.

In many instances, Theorem 2 gives no information since $h_i(A)$, $h_j(B)$ are often relatively prime. This, however, is not a failure of Theorem 2 because there are many instances in which the invariant factors of $A + B$ can be quite arbitrary even when those of A and B are fixed. For example, if $x|y$ and

$$A = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 \\ y & -1 \end{bmatrix},$$

then $h_1(A) = h_2(A) = 1$, $h_1(B) = h_2(B) = 1$, $h_1(C) = x$, $h_2(C) = y$, with x,y quite arbitrary. However, Theorem 2 does cover nontrivially the following sort of situation: If prime $p|h_1(A)$ and $p|h_1(B)$ then p must divide each element of A and B, therefore also each element of C, and so p necessarily is a divisor of $h_1(C)$.

The subscript $i + j - 1$ is the smallest for which Theorem 2 is true. Indeed, take $R = Z$, and set

$$A = I_{i-1} \dotplus (2) \dotplus 2I_{j-1} \dotplus 2I_{n-i-j+1},$$

$$B = -2I_{i-1} \dotplus (2) \dotplus -I_{j-1} \dotplus 2I_{n-1-j+1}$$

where I denotes an identity matrix and $\dotplus$ direct sum. Then $h_i(A) = 2$, $h_j(B) = 2$, but $h_{i+j-2}(C) = 1$.

References

1. M. Newman, *Integral matrices*, Academic Press, New York, 1972.
2. R. C. Thompson, *Smith invariants of products of integral matrices* (in preparation).
3. ______, *Interlacing inequalities for invariant factors*, Linear Algebra and Appl. **24** (1979), 1–31.

Institute for Algebra, Department of Mathematics, University of California at Santa Barbara, Santa Barbara, California 93106