INCIDENCE RINGS WITH SELF-DUALITY

JOEL K. HAACK

ABSTRACT. An artinian ring $R$ is said to have self-duality if there is a Morita duality between the categories of left and right finitely generated $R$-modules. Here it is shown that the incidence ring of a finite preordered set over a division ring has self-duality. This is accomplished in part by calculating their injective modules.

Theorems of Azumaya [3], Morita [11], and Tachikawa [13] in the late 1950's give necessary and sufficient conditions to insure that the category of finitely generated left modules over a (necessarily left artinian) ring $R$ is dual to the category of finitely generated right modules over a ring $S$. An open question in the subject of Morita duality for artinian rings is that of characterizing those artinian rings, other than artin algebras and QF rings, that have self-duality, that is, those for which there exists a duality between their categories of left and right finitely generated modules. Recently, artinian rings with self-duality have been shown to include certain factors of skew polynomial rings [12], hereditary artinian tensor rings satisfying the Dlab-Ringel duality conditions [2], rings with quivers that are trees [8] and many serial rings [9]. Here we prove that the incidence ring of a finite preordered set over a division ring (hereafter called a finite incidence ring) has self-duality. This class of rings properly contains the class of hereditary serial rings and, indeed, all artinian rings with quivers that are trees [8]. The self-duality constructed is weakly symmetric, so we may apply [9, (4.1)] to show that any factor ring of a finite incidence ring has a (weakly symmetric) self-duality.

A finite incidence ring over a division ring $D$ can be characterized as a (unital) subring of the $(n \times n)$-matrix ring over $D$ satisfying $R = \sum \{ DI_{kl} | I_{kk} R I_{ll} \neq 0 \}$, where $I_{kl}$ is the matrix unit with 1 in the $(k, l)$-position and 0 elsewhere. (Thus, finite incidence rings coincide with Mitchell's tic tac toe rings over division rings [10].) To prove that an artinian ring $R$ has self-duality, it is sufficient to show that the basic ring $eRe$ of $R$ is isomorphic to the endomorphism ring of the minimal left injective cogenerator over $eRe$ [3], [11]. It is not hard to show that the basic ring of a finite incidence ring is a finite incidence ring over the same division ring. Henceforth, we let $R$ be a basic indecomposable $(n \times n)$-finite incidence ring over the division ring $D$. We shall consider $D$ as the subring of constant diagonal matrices in $R$. Let $e_k = I_{kk}$ be the matrix unit with 1 in the $(k, k)$-position and 0 elsewhere. Then $e_k \in R$. If $I_{kl} \in R$, let $e_{kl} = I_{kl}$; if not, let $e_{kl} = 0$. The radical of $R$ is $J = J(R) = \sum \{ De_{ij} | i \neq j \}$. 

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We denote the composition length of a module $N$ by $c(N)$ and its injective envelope by $E(N)$. Let $M$ be the set of $(n \times 1)$-column vectors over $D$, with a left $R$-structure given by considering $R$ as a subring of the $(n \times n)$-matrix ring over $D$. Let $m_k = (\delta_{kj}) \in M$. Define a monomorphism $\iota_k: R e_k \to M$ by $\iota_k: r e_k \mapsto r e_k m_k = r m_k$. Since $c(e_k R e_k M) = 1$, we conclude from [4] that $M$ and hence also $R e_k$ are distributive (have a distributive lattice of submodules) for all $k$. Similarly, $e_k R$ is distributive for each $k$.

Now consider any submodule $N$ of $M$, and let $x \in N$ with $x = \sum_{k=1}^{n} d_k m_k$. If $d_k \neq 0$, then $e_k d_k^{-1} x = e_k d_k^{-1} d_k m_k = m_k$, so $m_k \in N$. Thus $x = \sum_{k=1}^{n} d_k m_k = \sum_{m_k \in N} d_k m_k$, so

$$N = \sum \{ Dm_k | m_k \in N \}.$$

Define submodules $L_k \subset M$ by $L_k = \sum \{ R m_j | e_k R e_j = 0 \}$. If $e_k R e_j \neq 0$, then $e_{kj} \neq 0$; thus $e_{kj} = 0$ implies $m_j \in L_k$. Conversely, suppose that $m_j \in L_k$ and write $m_j = \sum r_i m_i$ with $e_{ki} = 0$ and $r_i \in R$. Then $e_j m_j = \sum e_j r_i m_i$, so there exists $i$ with $e_j r_i e_i m_i \neq 0$ and $e_{ki} = 0$. Thus $e_j r_i e_i \neq 0$, so $e_{ji} \neq 0$. But now if $e_{kj} \neq 0$, then $0 \neq e_{kj} e_{ji}$ and $e_{ki} \neq 0$, a contradiction. Hence $m_j \in L_k$ implies $e_{kj} = 0$, and

$$L_k = \sum \{ Dm_j | e_{kj} = 0 \}.$$

Let $E_k = M / L_k$ and let $\eta_k$ be the natural epimorphism.

1. Proposition. The module $E_k = M / L_k$ is the injective envelope of $R e_k / J e_k$.

Proof. We first show that $\text{Soc}(E_k) \succeq R e_k / J e_k$. Let $x = \sum d_j m_j + L_k$ be any nonzero element of $E_k$ and suppose that $d_i \neq 0$ with $i \neq k$ and $e_{kj} \neq 0$. Then $0 \neq d_i m_i + L_k = e_{ki} \sum d_j m_j + L_k \in J e_k$. Therefore $x \in \text{Soc}(E_k)$. So $e_i \text{Soc}(E_k) = 0$ if $e_{ki} \neq 0$ and $k \neq i$, and of course $e_i \text{Soc}(E_k) = 0$ if $e_{ki} = 0$ (so that $m_i \in L_k$). Since $\text{Soc}(E_k) \neq 0$, we must have $e_i \text{Soc}(E_k) \neq 0$. Since $M$ is distributive, so also is $E_k$; thus $\text{Soc}(E_k) \succeq R e_k / J e_k$. To conclude we may apply [7, Lemma 5] and [6, Lemma 2.3] to see that the lattice of submodules of $E(R e_k / J e_k)$ is isomorphic to that of $e_k R$. Hence

$$c(E(R e_k / J e_k)) = c(e_k R) = \sum_{j=1}^{n} c(e_k R e_j R e_j)$$

$$= \# \{ j | e_k R e_j \neq 0 \} = n - \# \{ j | e_k R e_j = 0 \} = c(E_k).$$

Because $\text{Soc}(E_k) \succeq R e_k / J e_k$, $E_k$ is a submodule of $E(R e_k / J e_k)$ of the same length as $E(R e_k / J e_k)$, so $E_k = E(R e_k / J e_k)$.

Next we show that $\text{End}(M)$ is a division ring.

2. Proposition. Let $g: M \to M$.

(1) If $\ker g \neq 0$ then $g = 0$.

(2) If $\text{im} g \neq M$ then $g = 0$.

Thus $\text{End}(M)$ is a division ring. Moreover, $\text{End}(M) \simeq D$. 
Proof. If \( \ker g \neq 0 \), choose \( e_0 \) so that \( e_0 \ker g \neq 0 \), i.e., so that \( m_{e_0} \in \ker g \). Let \( m_h \) be given. We will show that \( m_h g = 0 \). Since \( R \) is indecomposable, there exist \( e_1, \ldots, e_l \) and \( e_1, \ldots, e_l = e_0 \) such that \( e_k \ker R \neq 0 = e_j \ker R \) for \( k = 1, \ldots, l \). (See [1, §7].) Suppose that \( m_{e_k} g = 0 \). Then also \( m_{e_k} g = 0 \), for if \( m_{e_k} g \neq 0 \), then \( 0 \neq e_k e_k (m_{e_k} g) = m_{e_k} g \). And now \( m_0 g = (e_{a_k} m_{e_0} g) = e_{a_k} (m_{e_k} g) = 0 \). Thus by induction \( m_h g = 0 \) for all \( h \) and \( M g = 0 \).

For (2), note that if \( \text{im} g \neq M \), then \( \ker g \neq 0 \), so by (1), \( g = 0 \).

For the moreover part, let \( f : M \to M \) and let \( S = Rm_h \) be a simple submodule of \( M \). Since \( M \) is distributive, \( \text{Soc}(M) \) is square-free so \( Sf \subset S \). Thus we may regard \( f |_S : S \to S \) as an \( e_k \ker R \) map. But then for some \( d \in D \), \( d_k m_k f |_S = d_k d m_k \) for all \( d_k m_k \in S \). Let \( f' : M \to M \) via \( f' : m \mapsto md \). Then \( 0 \neq S \subset \ker(f' - f) \), so by (1), \( f' = f \) and the ring monomorphism \( \Phi : D \to \text{End}(M) \) via \( \Phi(d) : m \mapsto md \) is onto. Hence \( D \cong \text{End}(M) \).

In our proof that finite incidence rings have self-duality, the main technique is that of changing the range or domain of a function. For example, if \( N \) is a distributive artinian module, \( L \) is a submodule of \( N \) and \( f \) is a map \( f : L \to N \), then \( \text{im} f \subset L \), so we may regard \( f \) as a map from \( L \) to \( N \). Dually, if \( N \) is a distributive noetherian module, \( L \) is a submodule of \( N \) and \( f \) is any map \( f : N \to N/L \), then \( \ker f \supset L \), so we may regard \( f \) as a map from \( N/L \) to \( N/L \) by the factor theorem. (See [5, §4.1].) These results allow us to develop the principal tool used in the proof of Theorem 4.

3. Lemma. (1) Let \( L \) be a nonzero indecomposable submodule of \( M \) and let \( f : L \to M \). Then there exists a unique map \( f' : M \to M \) such that \( If' = If \) for all \( l \in L \).

(2) Let \( L = M/K \) be a nonzero indecomposable factor of \( M \) and let \( f : M \to L \). Then there exists a unique map \( f' : M \to M \) such that \( mf' + K = mf \) for all \( m \in M \).

Proof. (1) Let \( L \) be a nonzero indecomposable submodule of \( M \) and let \( e = \sum (e_j e_j L \neq 0) \). Then \( c(eR eM) = 1 \) for all \( j \), either \( e_j L = 0 \) or \( e_j L = e_j M \). Thus \( L = eM \). Let \( f : L \to M \). By the remarks preceding the lemma, \( Lf \subset L \). Let \( f^* \) denote \( f \) with range restricted to \( L \). Now \( eR e \) is a finite incidence ring over \( D \), \( L = eM \) plays the role of \( M \) for \( eR e \) and \( f^* \in \text{End}(eR eM) \). Since \( eM \) is indecomposable over \( R \), it is an indecomposable module over \( eR e \). Since \( eM \) is also faithful over \( eR e \), \( eR e \) is an indecomposable ring and we may apply Proposition 2 to see that \( f^* \) is right multiplication by some \( d \in D \). Define \( f' : M \to M \) via \( f' : m \mapsto md \). Then \( f' \) extends \( f \). If \( g : M \to M \) also extends \( f \), then \( 0 \neq L \subset \ker(g - f') \), so by Proposition 2, \( g = f' \) and \( f' \) is unique.

(2) Let \( L = M/K \) be a nonzero indecomposable submodule of \( M \), let \( e = \sum (e_j e_j L \neq 0) \) and let \( f : M \to L \). By the remarks preceding the lemma, \( K \subset \ker f \), so we may define a map \( f^* : L \to L \) via \( f^* : m + K \mapsto mf \). Now \( eR e \) is a finite incidence ring over \( D \), \( L = e(M/K) \) plays the role of \( M \) for \( eR e \) and \( f^* \in \text{End}(eR e L) \). Since \( L \) is indecomposable over \( R \), it is an indecomposable module over \( eR e \). Since \( L \) is also faithful over \( eR e \), \( eR e \) is indecomposable and we may apply Proposition 2 to see that for some \( d \in D \), \( (m + K)f^* = md + K \) for all \( m \in M \). Define \( f' : M \to M \) via \( f' : m \mapsto md \). Then \( mf' + K = md + K = (m + K)f^* = mf \)
for all $m \in M$. If $g: M \to M$ also satisfies $mg + K = mf$ for all $m \in M$, then $m(g - f') \in K$ for all $m$, so $\text{im}(g - f') \neq M$. Thus by Proposition 2, $g = f'$.

A duality $D'$ between the categories of left and right finitely-generated $R$-modules is said to be weakly symmetric if for $J = \text{rad}(R)$ and for $e$ any primitive idempotent of $R$, $D'(Re/Je) \cong eR/eJ$. It is not hard to see that a ring isomorphism $\Phi: R \to \text{End}(Re)$ such that $E\Phi(e) = E(Re/Je)$, for each primitive idempotent $e \in R$, induces such a weakly symmetric duality if $R$ is artinian and $RE$ is an injective cogenerator [9, (3.1)].

4. THEOREM. Let $R$ be a finite incidence ring over a division ring. Then $R$ has a weakly symmetric duality.

PROOF. We may assume that $R$ is basic and indecomposable. Let the $R$-module $M$ be defined as it has been throughout. Let $\iota_i: Re_i \to M$ be the natural monomorphism and $\eta_i: M \to M/L_i = E_i$ be the natural epimorphism as before. Let $E = \bigoplus_{i=1}^n E_i$ be the minimal injective cogenerator of $R$-mod and let $S = \text{End}(RE)$ with $f_i \in S$ the natural projection onto $E_i$. Define $\theta: S \to R$ via $\theta: \sum f_isf_j \mapsto \sum e_ir_ej$, where $e_ir_ej$ is defined below.

Consider the following commutative diagram:

\[
\begin{array}{ccc}
Re_i & \xrightarrow{\iota_i} & M & \xrightarrow{\eta_i} & E_i \\
\downarrow e_ir_ej & \quad & \downarrow \delta & \quad & \downarrow f_isf_j \\
Re_j & \xrightarrow{\eta_j} & M & \xrightarrow{\iota_j} & E_j
\end{array}
\]

By Lemma 3 there exists a unique $\delta: M \to M$ such that $\eta_i f_isf_j = \delta \eta_j$. If $f_isf_j = 0$, then $\delta = 0$ and $e_ir_ej = 0$ is the only choice for $e_ir_ej$ to make the diagram commute. If $f_isf_j \neq 0$, then $\eta_i f_isf_j \neq 0$, so also $0 \neq e_ir_ej = \text{Hom}(Re_i, Re_j)$ [6, Theorem 2.4]. Thus $e_{rj} \neq 0$, so $m_{ij} = e_{ij} m_j$ and $m_{ij} \delta \subset Rm_i \subset Rm_j$. Therefore the map $e_ir_ej$ exists uniquely with $e_ir_ej \delta = e_j \delta$. Thus, $\theta$ is a well-defined function. To see that $\theta$ is bijective, let $e_{rj} \neq 0$ be given. By Lemma 3 there exists a unique $\delta: M \to M$ such that $e_ir_ej \delta = e_j \delta$. Now $e_ir_ej \neq 0$ implies that $e_{ij} \neq 0$, so that if $e_{jk} \neq 0$ then also $e_{k} = e_{j}e_{k} \neq 0$. Thus $e_{ik} = 0$ implies $e_{jk} = 0$, so $L_i \subset L_j$. Since $M$ is distributive, $L_i \delta \subset L_i \subset L_j \subset \text{Ker} \delta \eta_j$. Hence $\delta \eta_j$ factors uniquely through $\eta_j$. Let $\eta_j f_isf_j = \delta \eta_j$. Thus $\theta$ is bijective.

A simple argument shows that $\delta + \delta'$ and $e_ir_ej + e_ir'e_j$ are the maps associated with $f_isf_j + f_is's'f_j$, and it follows that $\theta$ is additive. Since for each $i, j, k \in \{1, \cdots, n\}$, $f_isf_j s'f_k$ corresponds to $\delta \delta'$ and then to $e_ir_ej r'e_k$, $\theta$ is multiplicative. Thus $\theta$ is a ring isomorphism. Also, $f_i$ corresponds to $1_M$ corresponds to $e_i$, so $\theta(f_i) = e_i$, and $R$ has a weakly symmetric duality.

If $R$ is an artinian ring with a weakly symmetric duality and the primitive right (or left) ideals of $R$ are distributive, then any factor ring of $R$ has a weakly symmetric duality [9, (4.1)]. Since the primitive right (and left) ideals of a finite incidence ring are distributive, we may apply Theorem 4 to conclude that factor rings of finite incidence rings also have self-duality.
5. **Corollary.** Any factor ring of a finite incidence ring over a division ring has a weakly symmetric duality.

We note here that a finite incidence ring over a division ring is a tensor ring iff it is hereditary, and an artinian tensor ring is a finite incidence ring iff every principal right and left ideal generated by a primitive idempotent is distributive. Nonhereditary finite incidence rings are nontrivial factors of tensor rings with the same quivers, but since such tensor rings do not satisfy the hypotheses of [9, (4.1)], one cannot apply a result of Auslander, Platzeck, and Reiten [2], namely, that hereditary artinian tensor rings satisfying the Dlab-Ringel duality conditions have self-duality, to show directly that finite incidence rings have self-duality.

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**References**


**Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242**

*Current address:* Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74074