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ALGEBRAIC AUTOMORPHISM GROUPS OF PRO-AFFINE ALGEBRAIC GROUPS

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ABSTRACT. We study the maximum connected algebraic subgroup of automorphisms of certain pro-affine algebraic groups.

Introduction. For a pro-affine algebraic group G over an algebraically closed field F of characteristic 0, let $W(G)$ denote the group of all automorphisms of G. Although the group $W(G)$ may not be given a pro-affine algebraic group structure, Hochschild in [4] introduced the notion of algebraic subgroups of $W(G)$ formulated in terms of the Hopf algebra of polynomial functions and showed, among other things, that if G is an affine algebraic group, then there is a connected algebraic subgroup of $W(G)$, which is the maximum in the sense that it contains every connected algebraic subgroup of $W(G)$. This result, however, does not extend to the pro-affine case as an example in [4] shows. We will call a pro-affine algebraic group G an (MC)-group if $W(G)$ contains the maximum connected algebraic subgroup.

In this paper, we consider the question of when a pro-affine algebraic group G is an (MC)-group. Our result (Theorem 1 in §4) states that if the unipotent radical of G is an (MC)-group, then so is G. This sharpens some of the results in [4]. Our study depends on the technique and results of [4] and also of [7]. We also show that an affine algebraic reductive group G over a field of characteristic 0 is conservative if and only if $\mathrm{Int}(G)$ is of finite index in $W(G)$ (Theorem 2 in §5). This generalizes Theorem 2.1 in [7].

1. General properties and notations. We begin by recalling some definitions and results from [4] and [5]. Let F be an algebraically closed field F of characteristic 0, and let G be a pro-affine algebraic group over F with Hopf algebra $\mathcal{Q}(G)$ of polynomial functions on G in the sense of [5]. We say that a subgroup P of $W(G)$ is *algebraic* if P can be given a pro-affine algebraic group structure over F so that the map $G \times P \rightarrow G$ sending each (x, α) of $G \times P$ onto $\alpha(x)$ is a morphism of pro-affine algebraic varieties. In this case, the Hopf algebra $\mathcal{Q}(P)$ of polynomial functions on P is generated (as an F-algebra) by the F-valued functions of the form ρ/f, with $\rho \in \mathrm{Hom}(\mathcal{Q}(G), F)$ and $f \in \mathcal{Q}(G)$, where ρ/f: $P \rightarrow F$ is given by

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$$(\rho/f)(\alpha) = \rho(f \circ \alpha) \quad \text{for } \alpha \in P.$$

If G is an affine (rather than pro-affine) group, then every algebraic subgroup of $W(G)$ is an affine algebraic group. We also know from [**4**, Theorem 2.1], that a subgroup P of $W(G)$ is contained in an algebraic subgroup of $W(G)$ if and only if $\mathcal{R}(G)$ is locally finite as a P-module, and if this condition is satisfied, the intersection $[P]$ of the family of all algebraic subgroups that contain P is the smallest algebraic subgroup of $W(G)$ in which P is algebraically dense. We will call $[P]$ the algebraic closure of P in $W(G)$.

In addition to the notations already introduced above, the following are standard throughout: Let G be a pro-affine algebraic group. Then G_0 will denote the connected component of 1 in G, and, for $x \in G$, I_x will denote the inner automorphism of G defined by $I_x(y) = xyx^{-1}$ for $y \in G$. For any subset A of G, let $\mathrm{Int}_G(A) = \{I_x \colon x \in A\}$ and we will simply write $\mathrm{Int}(G)$ for $\mathrm{Int}_G(G)$.

2. A lemma. Let G be a pro-affine algebraic group over an algebraically closed field F of characteristic 0, and let G_u be the unipotent radical of G. For a subgroup H of G, define $W(G)^H$ to be the subgroup of $W(G)$ consisting of $\alpha \in W(G)$ such that $\alpha(x) = x$ for all $x \in H$. Let $\zeta\colon W(G) \to W(G_u)$ denote the canonical map sending each $\alpha \in W(G)$ to the restriction $\alpha | G_u \in W(G_u)$.

LEMMA 1. *If Q is an algebraic subgroup of $W(G_u)$ and if K is a maximal reductive subgroup of G, then $\zeta^{-1}(Q) \cap W(G)^K$ is an algebraic subgroup of $W(G)$.*

PROOF. Put $P = \zeta^{-1}(Q) \cap W(G)^K$. Then $\mathcal{R}(G)$ is locally finite as a P-module. To see this we argue as in [**3**, p. 104]. Thus the semi-direct decomposition $G = G_u \cdot K$ induces the tensor product decomposition

$$\mathcal{R}(G) = \mathcal{R}(G)^K \otimes \mathcal{R}(G)^{G_u}$$

where $\mathcal{R}(G)^H$ for any subgroup H of G denotes the set of all H-fixed elements in $\mathcal{R}(G)$. Clearly P acts trivially on $\mathcal{R}(G)^{G_u}$ and P leaves $\mathcal{R}(G)^K$ invariant.

If we identify $\mathcal{R}(G)^K$ with $\mathcal{R}(G_u)$ via the restriction map $\mathcal{R}(G)^K \to \mathcal{R}(G_u)$ which is clearly an isomorphism, then the action of P on $\mathcal{R}(G)^K = \mathcal{R}(G_u)$ is the transpose of the natural action of $\zeta(P)$ on G_u. Since $\zeta(P) \subset Q$, and since the action of Q on $\mathcal{R}(G_u)$ is locally finite, it follows that the action of P on $\mathcal{R}(G_u)$ is locally finite, and this readily implies that $\mathcal{R}(G)$ is locally finite as a P-module.

Since $\zeta(P) \subset Q$ and since Q is an algebraic subgroup, ζ maps the algebraic closure $[P]$ of P into Q. Thus in order to show $[P] = P$, it is sufficient to show $[P] \subset W(G)^K$.

We first note that every element of $[P]$ leaves K invariant. In fact, let $k \in K$ and let $f \in \mathcal{R}(G)$ vanish on K. Then the polynomial function k/f: $[P] \to F$ maps P to $\{0\}$. Since P is algebraically dense in $[P]$, k/f maps $[P]$ to $\{0\}$. Thus if $\alpha \in [P]$, then $f(\alpha(k)) = (k/f)(\alpha) = 0$ and since this holds for an arbitrary $f \in \mathcal{A}(G)$ which vanishes on K, it follows that $\alpha(k) \in K$, proving that $\alpha(K) \subset K$.

The assignment of $\alpha \in [P]$ to $\alpha|K$ defines a map η: $[P] \to W(K)$. The image $\eta([P])$ is an algebraic subgroup of $W(K)$, and η induces a morphism $[P] \to \eta([P])$ of pro-affine algebraic groups (cf. [4, Proposition 2.2]). It follows that Ker η is an algebraic subgroup of $W(G)$. But Ker η is easily seen to be identical with P, so that P is an algebraic subgroup of $W(G)$.

3. Pro-affine reductive groups. We need the following lemma in §4.

LEMMA 2. *Let G be a reductive pro-affine algebraic group over an algebraically closed field F of characteristic* 0. *If P is any algebraic subgroup of $W(G)$, then P_0 is contained in* $\mathrm{Int}(G)$.

PROOF. Assume first that G is affine. We know from [4, Proposition 2.4] that $\mathrm{Int}(G)P$ is an algebraic subgroup of $W(G)$. Replacing P by $\mathrm{Int}(G)P$ if necessary, we may assume that $\mathrm{Int}(G)$ is contained in P.

We claim that the Lie algebra $\mathfrak{L}(P)$ of P may be identified with a subspace of the F-linear space $Z^1(G, \mathfrak{L}(G))$ consisting of all rational 1-cocycles of G with values in the Lie algebra $\mathfrak{L}(G)$ of G, on which G acts by the adjoint representation. Let $\sigma \in \mathfrak{L}(P)$. Thus σ is an F-linear map σ: $\mathcal{A}(P) \to F$ satisfying the usual differentiation condition.

For each $x \in G$, define σ_x: $\mathcal{A}(G) \to F$ by $\sigma_x(f) = \sigma(x/x^{-1} \cdot f)$, $f \in \mathcal{A}(G)$. Here $y \cdot f$ for any $y \in G$ denote the left translate of f by y, which is defined by $(y \cdot f)(z) = f(zy)$, $z \in G$. Then $\sigma_x \in \mathfrak{L}(G)$, and we have the relation $\sigma_{xy} = \sigma_x + \mathrm{Ad}(x)(\sigma_y)$ for $x, y \in G$. (See [7, pp. 146–148] for the detailed computation of the above and of others that follow below.) Thus the assignment $x \mapsto \sigma_x$ defines a cocycle $\sigma' \in Z^1(G, \mathfrak{L}(G))$. Since the functions x/f, together with their antipodes, generate $\mathcal{A}(P)$ as an F-algebra, it follows that the F-linear map $\sigma \mapsto \sigma'$ is an injection from $\mathfrak{L}(P)$ into $Z^1(G, \mathfrak{L}(G))$ under which we identify $\mathfrak{L}(P)$ with an F-subspace of $Z^1(G, \mathfrak{L}(G))$.

Consider now the morphism ν: $G \to P$ of affine groups which is given by $\nu(x) = I_x$, $x \in G$. The image of the differential $\mathfrak{L}(\nu)$: $\mathfrak{L}(G) \to \mathfrak{L}(P)$ of ν is exactly the F-subspace $B^1(G, \mathfrak{L}(G))$ of $Z^1(G, \mathfrak{L}(G))$ consisting of all 1-coboundaries of G. As G is reductive, the cohomology group $H^1(G, \mathfrak{L}(G))$ is trivial, and hence $\mathrm{Im}(\mathfrak{L}(\nu)) = B^1(G, \mathfrak{L}(g)) = Z^1(G, \mathfrak{L}(G))$ contains $\mathfrak{L}(P)$. Since F is algebraically closed, $\nu(G)$ is open in P, and consequently we have $P_0 \subset \nu(G) = \mathrm{Int}(G)$. This proves our assertion when G is affine.

Next we assume that G is an arbitrary reductive pro-affine algebraic group. Thus the Hopf algebra $\mathcal{A}(G)$ is a union of finitely generated sub-Hopf algebras B of $\mathcal{A}(G)$, and each of such B is in turn contained in a finitely generated P-stable sub-Hopf algebra of $\mathcal{A}(G)$ [4, Proposition 2.3]. For each finitely generated P-stable sub-Hopf algebra B of $\mathcal{A}(G)$, let G_B denote the affine algebraic group whose elements are the restrictions of those of $G = \mathrm{Hom}_{F\text{-alg}}(\mathcal{A}(G), F)$ to B. Then clearly G_B is reductive, and the canonical morphism π_B: $G \to G_B$ is surjective. For $\alpha \in P$, define $\alpha_B \in W(G_B)$ by the relation

$$\alpha_B(\pi_B(x)) = \pi_B(\alpha(x)) \quad \text{for all } x \in G.$$

The assignment $\alpha \mapsto \alpha_B$ defines a group homomorphism $\eta_B\colon P \to W(G_B)$, and $\eta_B(P)$ is an algebraic subgroup of $W(G_B)$. As we have already seen in the affine case, the identity component $\eta_B(P_0)$ is contained in $\mathrm{Int}(G_B)$. Now let $\alpha \in P_0$. Thus there exists $z \in G_B$ such that $\alpha_B(x') = zx'z^{-1}$ for all $x' \in G_B$. Consider the nonempty set $T(B) = \{z \in G_B\colon \alpha_B = I_z\}$. If z_1 and z_2 are elements of $T(B)$, then $z_1x'z_1^{-1} = z_2x'z_2^{-1}$ for all $x \in G_B$, which readily implies that $z_1^{-1}z_2$ is central in G_B. This shows that $T(B)$ is identical with a coset of the center of G_B, and as such is closed in G_B. Suppose that B_1 and B_2 are finitely generated P-stable sub-Hopf algebras of $\mathcal{Q}(G)$ with $B_1 \geqslant B_2$. The canonical morphism $G_{B_1} \to G_{B_2}$ is a closed map and maps $T(B_1)$ into $T(B_2)$. Hence the standard projective limit theorem (see, e.g., [**5**, p. 1131]) may be applied to the $T(B)$'s together with the closed maps $T(B_1) \to T(B_2)$ to conclude that the projective limit Q of the $T(B)$'s is nonempty. Now, let $y \in Q$. Then for all $x \in G$,

$$\pi_B(\alpha(x)) = \alpha_B(\pi_B(x)) = \pi_B(y)\pi_B(x)\pi_B(y)^{-1} = \pi_B(yxy^{-1})$$

for all B, and hence $\alpha(x) = yxy^{-1}$ for all $x \in G$, proving that $\alpha \in \mathrm{Int}(G)$.

4. Main theorem. For a pro-affine algebraic group G, the maximum connected algebraic subgroup of $W(G)$ (if it exists) is denoted by $W_1(G)$.

THEOREM 1. *Let G be a pro-affine algebraic group over an algebraically closed field F of characteristic* 0. *If the unipotent radical G_u is an* (MC)-*group, then so is G.*

PROOF. Choose a maximal reductive subgroup K of G, and let $A(K)$ denote the subgroup $\zeta^{-1}(W_1(G_u)) \cap W(G)^K$, using the notation of §2. By Lemma 1, $A(K)$ is an algebraic subgroup of $W(G)$. Thus our assertion will follow as soon as we show that every connected algebraic subgroup P of $W(G)$ is contained in $\mathrm{Int}_G(G_0)A(K)$.

Replacing P by the algebraic subgroup $\mathrm{Int}_G(G_0)P$ if necessary, we may assume that $\mathrm{Int}_G(G_0) \subset P$. Using the conjugacy theorem of maximal reductive subgroups [**3**, Theorem 14.2], we obtain $P = \mathrm{Int}_G(G_0) \cdot A$, where A is the subgroup of P consisting of all $\alpha \in P$ leaving K invariant.

Let $\eta\colon W(G) \to W(K)$ be the map obtained by composing the canonical map $W(G) \to W(G/G_u)$ with an isomorphism $W(G/G_u) \cong W(K)$ resulting from an isomorphism $G/G_u \cong K$. Then $\eta(P)$ is a connected algebraic subgroup of $W(K)$. Since $W_1(K) = \mathrm{Int}_K(K_0)$ by Lemma 2, $\eta(P)$ (and hence $\eta(A)$) is contained in $\mathrm{Int}_K(K_0)$. On the other hand, $\eta(\alpha) = \alpha|K$ for $\alpha \in A$, and η maps the subgroup $\mathrm{Int}_G(K_0)$ of A onto $\mathrm{Int}_K(K_0)$. We can therefore write $A = \mathrm{Int}_G(K_0)\mathrm{Ker}(\eta|A)$, and consequently $P = \mathrm{Int}_G(G_0)\mathrm{Ker}(\eta|A)$. Clearly $\mathrm{Ker}(\eta|A) \subset A(K)$, and we have $P \leqslant \mathrm{Int}_G(G_0)A(K)$, proving our assertion.

REMARK. It is clear that under the hypothesis of Theorem 1, $W_1(G) = \mathrm{Int}_G(G_0)A(K)_0$.

COROLLARY 1. *Let G be as in the theorem. If G_u is finite dimensional, then G is an* (MC)-*group. In particular, every affine algebraic group is an* (MC)-*group.*

PROOF. If G_u is finite dimensional, then $W(G_u)$ is an affine algebraic group, and is, in fact, isomorphic with the algebraic group $W(L)$ of all automorphisms of the Lie algebra L of G_u. Thus G_u is an (MC)-group, and G is (MC) by the theorem.

REMARK. If G is as in Corollary 1, we can describe $W_1(G)$ fairly easily. Choose a maximal reductive algebraic subgroup K of G. Applying Lemma 1 to $Q = W(G_u)$, we see that $W(G)^K$ is an algebraic subgroup of $W(G)$. It is also clear from the proof of Theorem 1 that every connected algebraic subgroup of $W(G)$ is contained in the algebraic subgroup $\mathrm{Int}_G(G_0)W(G)^K$. It follows that $W_1(G)$ is equal to $\mathrm{Int}_G(G_0)(W(G)^K)_0$, the identity component of $\mathrm{Int}_G(G_0)W(G)^K$. The above description of $W_1(G)$ is given in **[4]** (see the proof of Theorem 4.2) when G is a connected affine algebraic group.

COROLLARY 2. *Let G be a connected pro-affine algebraic group over F. Then any normal algebraic subgroup N, which is either pro-finite or reductive and commutative, is central in G.*

PROOF. In both cases, $W_1(N) = \{1\}$ by Lemma 2. Define $\rho: G \to W(N)$ by $\rho(x)(n) = xnx^{-1}$, $n \in N$ and $x \in G$. Then $\rho(G)$ is a connected algebraic subgroup of $W(N)$, so that $\rho(G) = \{1\}$, which implies that N is central in G.

5. Conservative reductive groups. We recall from **[6]** that an affine algebraic group G over a field F of characteristic 0 is said to be conservative if $\mathcal{Q}(G)$ is locally finite as a $W(G)$-module, and that if G is conservative then the automorphism group $W(G)$ itself becomes an affine algebraic group in a natural way.

The following theorem was proved in **[7]** when F is algebraically closed.

THEOREM 2. *Let G be a reductive affine algebraic group over a field F of characteristic* 0. *Then G is conservative if and only if* $\mathrm{Int}(G)$ *is of finite index in* $W(G)$.

PROOF. Assume F is not algebraically closed, and let L be an algebraically closed field containing F as a subfield. Consider the affine algebraic group G^L over L obtained from G by extending the field F to L. Then $\mathfrak{L}(G^L) = \mathfrak{L}(G) \otimes_F L$.

Let Ad: $G \to Gl_F(\mathfrak{L}(G))$ (resp. Ad$'$: $G^L \to Gl_L(\mathfrak{L}(G^L))$) denote the adjoint representation of G (resp. of G^L). Then $Gl_F(\mathfrak{L}(G))^L = Gl_L(\mathfrak{L}(G^L))$, and by a result of Chevalley **[2**, p. 109, Proposition 4], $\overline{\mathrm{Ad}(G)}^L = \mathrm{Ad}'(G^L)$, where $\overline{\mathrm{Ad}(G)}$ denotes the closure of $\mathrm{Ad}(G)$ in the algebraic group $Gl_F(\mathfrak{L}(G))$. This shows that $\overline{\mathrm{Ad}(G)}$ is the set of all F-rational points of $\mathrm{Ad}'(G^L)$.

On the other hand, $\mathrm{Ad}'(G^L)$ is the orbit of the affine group G^L at

$1 \in Gl_L(\mathfrak{L}(G^L))$, where G^L is viewed as a transformation group acting on the affine variety $Gl_L(\mathfrak{L}(G^L))$ via the adjoint representation Ad′. That is, $\mathrm{Ad}'(G^L)$ is identified with a homogeneous space of G^L. By [1, Corollary 6.4], the set $\overline{\mathrm{Ad}(G)}$ of F-rational points of $\mathrm{Ad}'(G^L)$ is a union of finitely many orbits of G. It follows that $\overline{\mathrm{Ad}(G)}/\mathrm{Ad}(G)$, and hence $\overline{\mathrm{Int}(G)}/\mathrm{Int}(G)$, is finite.

Suppose now that G is conservative. Then the map $\nu: G \to W(G)$, given by $\nu(x) = I_x$, $x \in G$, a morphism of affine algebraic groups, and, as we have seen in the proof of Theorem 2.1 in [7], the differential $\mathfrak{L}(\nu)$: $\mathfrak{L}(G) \to \mathfrak{L}(W(G))$ is surjective. Since F is of characteristic 0, the closure of $\nu(G) = \mathrm{Int}(G)$ is open in $W(G)$, and hence $\overline{\mathrm{Int}(G)}$ is of finite index in $W(G)$. Since $\mathrm{Int}(G)$ is of finite index in $\overline{\mathrm{Int}(G)}$, it follows that $\mathrm{Int}(G)$ is of finite index in $W(G)$.

The other implication in the theorem is clear from the fact that $\mathfrak{A}(G)$ is a locally finite $\mathrm{Int}(G)$-module and that $\mathrm{Int}(G)$ is a normal subgroup of $W(G)$.

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