

## ALGEBRAIC AUTOMORPHISM GROUPS OF PRO-AFFINE ALGEBRAIC GROUPS

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**ABSTRACT.** We study the maximum connected algebraic subgroup of automorphisms of certain pro-affine algebraic groups.

**Introduction.** For a pro-affine algebraic group  $G$  over an algebraically closed field  $F$  of characteristic 0, let  $W(G)$  denote the group of all automorphisms of  $G$ . Although the group  $W(G)$  may not be given a pro-affine algebraic group structure, Hochschild in [4] introduced the notion of algebraic subgroups of  $W(G)$  formulated in terms of the Hopf algebra of polynomial functions and showed, among other things, that if  $G$  is an affine algebraic group, then there is a connected algebraic subgroup of  $W(G)$ , which is the maximum in the sense that it contains every connected algebraic subgroup of  $W(G)$ . This result, however, does not extend to the pro-affine case as an example in [4] shows. We will call a pro-affine algebraic group  $G$  an (MC)-group if  $W(G)$  contains the maximum connected algebraic subgroup.

In this paper, we consider the question of when a pro-affine algebraic group  $G$  is an (MC)-group. Our result (Theorem 1 in §4) states that if the unipotent radical of  $G$  is an (MC)-group, then so is  $G$ . This sharpens some of the results in [4]. Our study depends on the technique and results of [4] and also of [7]. We also show that an affine algebraic reductive group  $G$  over a field of characteristic 0 is conservative if and only if  $\text{Int}(G)$  is of finite index in  $W(G)$  (Theorem 2 in §5). This generalizes Theorem 2.1 in [7].

**1. General properties and notations.** We begin by recalling some definitions and results from [4] and [5]. Let  $F$  be an algebraically closed field of characteristic 0, and let  $G$  be a pro-affine algebraic group over  $F$  with Hopf algebra  $\mathcal{Q}(G)$  of polynomial functions on  $G$  in the sense of [5]. We say that a subgroup  $P$  of  $W(G)$  is *algebraic* if  $P$  can be given a pro-affine algebraic group structure over  $F$  so that the map  $G \times P \rightarrow G$  sending each  $(x, \alpha)$  of  $G \times P$  onto  $\alpha(x)$  is a morphism of pro-affine algebraic varieties. In this case, the Hopf algebra  $\mathcal{Q}(P)$  of polynomial functions on  $P$  is generated (as an  $F$ -algebra) by the  $F$ -valued functions of the form  $\rho/f$ , with  $\rho \in \text{Hom}(\mathcal{Q}(G), F)$  and  $f \in \mathcal{Q}(G)$ , where  $\rho/f: P \rightarrow F$  is given by

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$$(\rho/f)(\alpha) = \rho(f \circ \alpha) \quad \text{for } \alpha \in P.$$

If  $G$  is an affine (rather than pro-affine) group, then every algebraic subgroup of  $W(G)$  is an affine algebraic group. We also know from [4, Theorem 2.1], that a subgroup  $P$  of  $W(G)$  is contained in an algebraic subgroup of  $W(G)$  if and only if  $\mathcal{Q}(G)$  is locally finite as a  $P$ -module, and if this condition is satisfied, the intersection  $[P]$  of the family of all algebraic subgroups that contain  $P$  is the smallest algebraic subgroup of  $W(G)$  in which  $P$  is algebraically dense. We will call  $[P]$  the algebraic closure of  $P$  in  $W(G)$ .

In addition to the notations already introduced above, the following are standard throughout: Let  $G$  be a pro-affine algebraic group. Then  $G_0$  will denote the connected component of 1 in  $G$ , and, for  $x \in G$ ,  $I_x$  will denote the inner automorphism of  $G$  defined by  $I_x(y) = xyx^{-1}$  for  $y \in G$ . For any subset  $A$  of  $G$ , let  $\text{Int}_G(A) = \{I_x : x \in A\}$  and we will simply write  $\text{Int}(G)$  for  $\text{Int}_G(G)$ .

**2. A lemma.** Let  $G$  be a pro-affine algebraic group over an algebraically closed field  $F$  of characteristic 0, and let  $G_u$  be the unipotent radical of  $G$ . For a subgroup  $H$  of  $G$ , define  $W(G)^H$  to be the subgroup of  $W(G)$  consisting of  $\alpha \in W(G)$  such that  $\alpha(x) = x$  for all  $x \in H$ . Let  $\zeta: W(G) \rightarrow W(G_u)$  denote the canonical map sending each  $\alpha \in W(G)$  to the restriction  $\alpha|_{G_u} \in W(G_u)$ .

**LEMMA 1.** *If  $Q$  is an algebraic subgroup of  $W(G_u)$  and if  $K$  is a maximal reductive subgroup of  $G$ , then  $\zeta^{-1}(Q) \cap W(G)^K$  is an algebraic subgroup of  $W(G)$ .*

**PROOF.** Put  $P = \zeta^{-1}(Q) \cap W(G)^K$ . Then  $\mathcal{Q}(G)$  is locally finite as a  $P$ -module. To see this we argue as in [3, p. 104]. Thus the semi-direct decomposition  $G = G_u \cdot K$  induces the tensor product decomposition

$$\mathcal{Q}(G) = \mathcal{Q}(G)^K \otimes \mathcal{Q}(G)^{G_u}$$

where  $\mathcal{Q}(G)^H$  for any subgroup  $H$  of  $G$  denotes the set of all  $H$ -fixed elements in  $\mathcal{Q}(G)$ . Clearly  $P$  acts trivially on  $\mathcal{Q}(G)^{G_u}$  and  $P$  leaves  $\mathcal{Q}(G)^K$  invariant.

If we identify  $\mathcal{Q}(G)^K$  with  $\mathcal{Q}(G_u)$  via the restriction map  $\mathcal{Q}(G)^K \rightarrow \mathcal{Q}(G_u)$  which is clearly an isomorphism, then the action of  $P$  on  $\mathcal{Q}(G)^K = \mathcal{Q}(G_u)$  is the transpose of the natural action of  $\zeta(P)$  on  $G_u$ . Since  $\zeta(P) \subset Q$ , and since the action of  $Q$  on  $\mathcal{Q}(G_u)$  is locally finite, it follows that the action of  $P$  on  $\mathcal{Q}(G_u)$  is locally finite, and this readily implies that  $\mathcal{Q}(G)$  is locally finite as a  $P$ -module.

Since  $\zeta(P) \subset Q$  and since  $Q$  is an algebraic subgroup,  $\zeta$  maps the algebraic closure  $[P]$  of  $P$  into  $Q$ . Thus in order to show  $[P] = P$ , it is sufficient to show  $[P] \subset W(G)^K$ .

We first note that every element of  $[P]$  leaves  $K$  invariant. In fact, let  $k \in K$  and let  $f \in \mathcal{Q}(G)$  vanish on  $K$ . Then the polynomial function  $k/f: [P] \rightarrow F$  maps  $P$  to  $\{0\}$ . Since  $P$  is algebraically dense in  $[P]$ ,  $k/f$  maps  $[P]$  to  $\{0\}$ . Thus if  $\alpha \in [P]$ , then  $f(\alpha(k)) = (k/f)(\alpha) = 0$  and since this holds for an

arbitrary  $f \in \mathcal{Q}(G)$  which vanishes on  $K$ , it follows that  $\alpha(k) \in K$ , proving that  $\alpha(K) \subset K$ .

The assignment of  $\alpha \in [P]$  to  $\alpha|_K$  defines a map  $\eta: [P] \rightarrow W(K)$ . The image  $\eta([P])$  is an algebraic subgroup of  $W(K)$ , and  $\eta$  induces a morphism  $[P] \rightarrow \eta([P])$  of pro-affine algebraic groups (cf. [4, Proposition 2.2]). It follows that  $\text{Ker } \eta$  is an algebraic subgroup of  $W(G)$ . But  $\text{Ker } \eta$  is easily seen to be identical with  $P$ , so that  $P$  is an algebraic subgroup of  $W(G)$ .

**3. Pro-affine reductive groups.** We need the following lemma in §4.

**LEMMA 2.** *Let  $G$  be a reductive pro-affine algebraic group over an algebraically closed field  $F$  of characteristic 0. If  $P$  is any algebraic subgroup of  $W(G)$ , then  $P_0$  is contained in  $\text{Int}(G)$ .*

**PROOF.** Assume first that  $G$  is affine. We know from [4, Proposition 2.4] that  $\text{Int}(G)P$  is an algebraic subgroup of  $W(G)$ . Replacing  $P$  by  $\text{Int}(G)P$  if necessary, we may assume that  $\text{Int}(G)$  is contained in  $P$ .

We claim that the Lie algebra  $\mathcal{L}(P)$  of  $P$  may be identified with a subspace of the  $F$ -linear space  $Z^1(G, \mathcal{L}(G))$  consisting of all rational 1-cocycles of  $G$  with values in the Lie algebra  $\mathcal{L}(G)$  of  $G$ , on which  $G$  acts by the adjoint representation. Let  $\sigma \in \mathcal{L}(P)$ . Thus  $\sigma$  is an  $F$ -linear map  $\sigma: \mathcal{Q}(P) \rightarrow F$  satisfying the usual differentiation condition.

For each  $x \in G$ , define  $\sigma_x: \mathcal{Q}(G) \rightarrow F$  by  $\sigma_x(f) = \sigma(x/x^{-1} \cdot f)$ ,  $f \in \mathcal{Q}(G)$ . Here  $y \cdot f$  for any  $y \in G$  denote the left translate of  $f$  by  $y$ , which is defined by  $(y \cdot f)(z) = f(yz)$ ,  $z \in G$ . Then  $\sigma_x \in \mathcal{L}(G)$ , and we have the relation  $\sigma_{xy} = \sigma_x + \text{Ad}(x)(\sigma_y)$  for  $x, y \in G$ . (See [7, pp. 146–148] for the detailed computation of the above and of others that follow below.) Thus the assignment  $x \mapsto \sigma_x$  defines a cocycle  $\sigma' \in Z^1(G, \mathcal{L}(G))$ . Since the functions  $x/f$ , together with their antipodes, generate  $\mathcal{Q}(P)$  as an  $F$ -algebra, it follows that the  $F$ -linear map  $\sigma \mapsto \sigma'$  is an injection from  $\mathcal{L}(P)$  into  $Z^1(G, \mathcal{L}(G))$  under which we identify  $\mathcal{L}(P)$  with an  $F$ -subspace of  $Z^1(G, \mathcal{L}(G))$ .

Consider now the morphism  $\nu: G \rightarrow P$  of affine groups which is given by  $\nu(x) = I_x$ ,  $x \in G$ . The image of the differential  $\mathcal{L}(\nu): \mathcal{L}(G) \rightarrow \mathcal{L}(P)$  of  $\nu$  is exactly the  $F$ -subspace  $B^1(G, \mathcal{L}(G))$  of  $Z^1(G, \mathcal{L}(G))$  consisting of all 1-coboundaries of  $G$ . As  $G$  is reductive, the cohomology group  $H^1(G, \mathcal{L}(G))$  is trivial, and hence  $\text{Im}(\mathcal{L}(\nu)) = B^1(G, \mathcal{L}(G)) = Z^1(G, \mathcal{L}(G))$  contains  $\mathcal{L}(P)$ . Since  $F$  is algebraically closed,  $\nu(G)$  is open in  $P$ , and consequently we have  $P_0 \subset \nu(G) = \text{Int}(G)$ . This proves our assertion when  $G$  is affine.

Next we assume that  $G$  is an arbitrary reductive pro-affine algebraic group. Thus the Hopf algebra  $\mathcal{Q}(G)$  is a union of finitely generated sub-Hopf algebras  $B$  of  $\mathcal{Q}(G)$ , and each of such  $B$  is in turn contained in a finitely generated  $P$ -stable sub-Hopf algebra of  $\mathcal{Q}(G)$  [4, Proposition 2.3]. For each finitely generated  $P$ -stable sub-Hopf algebra  $B$  of  $\mathcal{Q}(G)$ , let  $G_B$  denote the affine algebraic group whose elements are the restrictions of those of  $G = \text{Hom}_{F\text{-alg}}(\mathcal{Q}(G), F)$  to  $B$ . Then clearly  $G_B$  is reductive, and the canonical morphism  $\pi_B: G \rightarrow G_B$  is surjective. For  $\alpha \in P$ , define  $\alpha_B \in W(G_B)$  by the

relation

$$\alpha_B(\pi_B(x)) = \pi_B(\alpha(x)) \quad \text{for all } x \in G.$$

The assignment  $\alpha \mapsto \alpha_B$  defines a group homomorphism  $\eta_B: P \rightarrow W(G_B)$ , and  $\eta_B(P)$  is an algebraic subgroup of  $W(G_B)$ . As we have already seen in the affine case, the identity component  $\eta_B(P_0)$  is contained in  $\text{Int}(G_B)$ . Now let  $\alpha \in P_0$ . Thus there exists  $z \in G_B$  such that  $\alpha_B(x') = zx'z^{-1}$  for all  $x' \in G_B$ . Consider the nonempty set  $T(B) = \{z \in G_B: \alpha_B = I_z\}$ . If  $z_1$  and  $z_2$  are elements of  $T(B)$ , then  $z_1x'z_1^{-1} = z_2x'z_2^{-1}$  for all  $x \in G_B$ , which readily implies that  $z_1^{-1}z_2$  is central in  $G_B$ . This shows that  $T(B)$  is identical with a coset of the center of  $G_B$ , and as such is closed in  $G_B$ . Suppose that  $B_1$  and  $B_2$  are finitely generated  $P$ -stable sub-Hopf algebras of  $\mathcal{Q}(G)$  with  $B_1 \supseteq B_2$ . The canonical morphism  $G_{B_1} \rightarrow G_{B_2}$  is a closed map and maps  $T(B_1)$  into  $T(B_2)$ . Hence the standard projective limit theorem (see, e.g., [5, p. 1131]) may be applied to the  $T(B)$ 's together with the closed maps  $T(B_1) \rightarrow T(B_2)$  to conclude that the projective limit  $Q$  of the  $T(B)$ 's is nonempty. Now, let  $y \in Q$ . Then for all  $x \in G$ ,

$$\pi_B(\alpha(x)) = \alpha_B(\pi_B(x)) = \pi_B(y)\pi_B(x)\pi_B(y)^{-1} = \pi_B(yxy^{-1})$$

for all  $B$ , and hence  $\alpha(x) = yxy^{-1}$  for all  $x \in G$ , proving that  $\alpha \in \text{Int}(G)$ .

**4. Main theorem.** For a pro-affine algebraic group  $G$ , the maximum connected algebraic subgroup of  $W(G)$  (if it exists) is denoted by  $W_1(G)$ .

**THEOREM 1.** *Let  $G$  be a pro-affine algebraic group over an algebraically closed field  $F$  of characteristic 0. If the unipotent radical  $G_u$  is an (MC)-group, then so is  $G$ .*

**PROOF.** Choose a maximal reductive subgroup  $K$  of  $G$ , and let  $A(K)$  denote the subgroup  $\zeta^{-1}(W_1(G_u)) \cap W(G)^K$ , using the notation of §2. By Lemma 1,  $A(K)$  is an algebraic subgroup of  $W(G)$ . Thus our assertion will follow as soon as we show that every connected algebraic subgroup  $P$  of  $W(G)$  is contained in  $\text{Int}_G(G_0)A(K)$ .

Replacing  $P$  by the algebraic subgroup  $\text{Int}_G(G_0)P$  if necessary, we may assume that  $\text{Int}_G(G_0) \subset P$ . Using the conjugacy theorem of maximal reductive subgroups [3, Theorem 14.2], we obtain  $P = \text{Int}_G(G_0) \cdot A$ , where  $A$  is the subgroup of  $P$  consisting of all  $\alpha \in P$  leaving  $K$  invariant.

Let  $\eta: W(G) \rightarrow W(K)$  be the map obtained by composing the canonical map  $W(G) \rightarrow W(G/G_u)$  with an isomorphism  $W(G/G_u) \cong W(K)$  resulting from an isomorphism  $G/G_u \cong K$ . Then  $\eta(P)$  is a connected algebraic subgroup of  $W(K)$ . Since  $W_1(K) = \text{Int}_K(K_0)$  by Lemma 2,  $\eta(P)$  (and hence  $\eta(A)$ ) is contained in  $\text{Int}_K(K_0)$ . On the other hand,  $\eta(\alpha) = \alpha|K$  for  $\alpha \in A$ , and  $\eta$  maps the subgroup  $\text{Int}_G(K_0)$  of  $A$  onto  $\text{Int}_K(K_0)$ . We can therefore write  $A = \text{Int}_G(K_0)\text{Ker}(\eta|A)$ , and consequently  $P = \text{Int}_G(G_0)\text{Ker}(\eta|A)$ . Clearly  $\text{Ker}(\eta|A) \subset A(K)$ , and we have  $P \leq \text{Int}_G(G_0)A(K)$ , proving our assertion.

REMARK. It is clear that under the hypothesis of Theorem 1,  $W_1(G) = \text{Int}_G(G_0)A(K)_0$ .

COROLLARY 1. *Let  $G$  be as in the theorem. If  $G_u$  is finite dimensional, then  $G$  is an (MC)-group. In particular, every affine algebraic group is an (MC)-group.*

PROOF. If  $G_u$  is finite dimensional, then  $W(G_u)$  is an affine algebraic group, and is, in fact, isomorphic with the algebraic group  $W(L)$  of all automorphisms of the Lie algebra  $L$  of  $G_u$ . Thus  $G_u$  is an (MC)-group, and  $G$  is (MC) by the theorem.

REMARK. If  $G$  is as in Corollary 1, we can describe  $W_1(G)$  fairly easily. Choose a maximal reductive algebraic subgroup  $K$  of  $G$ . Applying Lemma 1 to  $Q = W(G_u)$ , we see that  $W(G)^K$  is an algebraic subgroup of  $W(G)$ . It is also clear from the proof of Theorem 1 that every connected algebraic subgroup of  $W(G)$  is contained in the algebraic subgroup  $\text{Int}_G(G_0)W(G)^K$ . It follows that  $W_1(G)$  is equal to  $\text{Int}_G(G_0)(W(G)^K)_0$ , the identity component of  $\text{Int}_G(G_0)W(G)^K$ . The above description of  $W_1(G)$  is given in [4] (see the proof of Theorem 4.2) when  $G$  is a connected affine algebraic group.

COROLLARY 2. *Let  $G$  be a connected pro-affine algebraic group over  $F$ . Then any normal algebraic subgroup  $N$ , which is either pro-finite or reductive and commutative, is central in  $G$ .*

PROOF. In both cases,  $W_1(N) = \{1\}$  by Lemma 2. Define  $\rho: G \rightarrow W(N)$  by  $\rho(x)(n) = xnx^{-1}$ ,  $n \in N$  and  $x \in G$ . Then  $\rho(G)$  is a connected algebraic subgroup of  $W(N)$ , so that  $\rho(G) = \{1\}$ , which implies that  $N$  is central in  $G$ .

**5. Conservative reductive groups.** We recall from [6] that an affine algebraic group  $G$  over a field  $F$  of characteristic 0 is said to be conservative if  $\mathcal{A}(G)$  is locally finite as a  $W(G)$ -module, and that if  $G$  is conservative then the automorphism group  $W(G)$  itself becomes an affine algebraic group in a natural way.

The following theorem was proved in [7] when  $F$  is algebraically closed.

THEOREM 2. *Let  $G$  be a reductive affine algebraic group over a field  $F$  of characteristic 0. Then  $G$  is conservative if and only if  $\text{Int}(G)$  is of finite index in  $W(G)$ .*

PROOF. Assume  $F$  is not algebraically closed, and let  $L$  be an algebraically closed field containing  $F$  as a subfield. Consider the affine algebraic group  $G^L$  over  $L$  obtained from  $G$  by extending the field  $F$  to  $L$ . Then  $\mathcal{L}(G^L) = \mathcal{L}(G) \otimes_F L$ .

Let  $\text{Ad}: G \rightarrow \text{Gl}_F(\mathcal{L}(G))$  (resp.  $\text{Ad}': G^L \rightarrow \text{Gl}_L(\mathcal{L}(G^L))$ ) denote the adjoint representation of  $G$  (resp. of  $G^L$ ). Then  $\text{Gl}_F(\mathcal{L}(G))^L = \text{Gl}_L(\mathcal{L}(G^L))$ , and by a result of Chevalley [2, p. 109, Proposition 4],  $\overline{\text{Ad}(G)}^L = \text{Ad}'(G^L)$ , where  $\overline{\text{Ad}(G)}$  denotes the closure of  $\text{Ad}(G)$  in the algebraic group  $\text{Gl}_F(\mathcal{L}(G))$ . This shows that  $\overline{\text{Ad}(G)}$  is the set of all  $F$ -rational points of  $\text{Ad}'(G^L)$ .

On the other hand,  $\text{Ad}'(G^L)$  is the orbit of the affine group  $G^L$  at

$1 \in Gl_L(\mathcal{L}(G^L))$ , where  $G^L$  is viewed as a transformation group acting on the affine variety  $Gl_L(\mathcal{L}(G^L))$  via the adjoint representation  $\text{Ad}'$ . That is,  $\text{Ad}'(G^L)$  is identified with a homogeneous space of  $G^L$ . By [1, Corollary 6.4], the set  $\overline{\text{Ad}(G)}$  of  $F$ -rational points of  $\text{Ad}'(G^L)$  is a union of finitely many orbits of  $G$ . It follows that  $\overline{\text{Ad}(G)}/\text{Ad}(G)$ , and hence  $\overline{\text{Int}(G)}/\text{Int}(G)$ , is finite.

Suppose now that  $G$  is conservative. Then the map  $\nu: G \rightarrow W(G)$ , given by  $\nu(x) = I_x$ ,  $x \in G$ , a morphism of affine algebraic groups, and, as we have seen in the proof of Theorem 2.1 in [7], the differential  $\mathcal{L}(\nu): \mathcal{L}(G) \rightarrow \mathcal{L}(W(G))$  is surjective. Since  $F$  is of characteristic 0, the closure of  $\nu(G) = \text{Int}(G)$  is open in  $W(G)$ , and hence  $\overline{\text{Int}(G)}$  is of finite index in  $W(G)$ . Since  $\text{Int}(G)$  is of finite index in  $\overline{\text{Int}(G)}$ , it follows that  $\text{Int}(G)$  is of finite index in  $W(G)$ .

The other implication in the theorem is clear from the fact that  $\mathcal{Q}(G)$  is a locally finite  $\text{Int}(G)$ -module and that  $\text{Int}(G)$  is a normal subgroup of  $W(G)$ .

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