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ON RESIDUAL PROPERTIES

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ABSTRACT. It is shown that the class of structures residually in a universal class closed under elementary equivalence is definable by a sentence of $L_{\infty\omega}$.

In [2] and [3], Neumann showed the class of residually finite groups is of countable character; that is, a group is residually finite if every countable subgroup is. Sabbagh [4] gave a simple proof using ultraproducts of a generalization of this result. Here we use the completeness theorem to obtain a further generalization of this result. In addition we prove a definability theorem which implies the class of residually finite groups is definable by a sentence of $L_{\omega_1\omega}$. This answers a question raised in [4].

In the following we associate a finitary language with a class of structures in the normal manner and assume that similarity types match where necessary. We begin with some definitions. Suppose X is a class of structures. A structure A is *residually in* X (denoted $A \in RX$) if for any two elements $a, b \in A$ with $a \neq b$ there is $C \in X$ and an onto homomorphism f: $A \to C$ such that $f(a) \neq f(b)$. A class of structures, X, is *universal* if each substructure of X is itself an element of X. (Note X is not necessarily an elementary class.) A theory T is *universal* if the class of models of T is universal. It is possible (cf. [1, Theorem 3.2.2, p. 124]) to give a syntactic characterization of universal theories. If A is a structure, $D^+(A)$ will denote the set of atomic sentences, in the language with constants added for elements of A, true in A. (If A is a group, $D^+(A)$ is the multiplication table.)

The following is an immediate consequence of the completeness theorem.

LEMMA. *Suppose X is the class of models of a universal theory T, and $a, b \in A$. There exists an onto homomorphism f: $A \to C \in X$ with $f(a) \neq f(b)$ if and only if $T \cup D^+(A) \cup \{\neg(a = b)\}$ is consistent.*

If T is a set of sentences let $\Sigma(T) = \{\varphi(x_0, x_1, y_0, \ldots, y_t) | \varphi(x_0, x_1, y_0, \ldots, y_t) \to \neg\psi$ for some $\psi \in T$ and $\varphi(x_0, x_1, y_0, \ldots, y_t)$ is the conjunction of atomic formulae and $\neg(x_0 = x_1)\}$. From the lemma we have:

THEOREM 1. *If X and T are as in the Lemma, the class RX is axiomatized by*

$$\{\forall x_0, x_1, y_0, \ldots, y_t \neg\varphi(x_0, x_1, y_0, \ldots, y_t) | \varphi(x_0, x_1, y_0, \ldots, y_t) \in \Sigma(T)\}.$$

We have, as well, the promised definability theorem.

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Theorem 2. *If X is universal and closed under elementary equivalence, then the class RX can be axiomatized by a sentence in $L_{\infty\omega}$.*

Proof. By the hypothesis on X we can choose X_i, T_i ($i \in I$, an index set) as in the Lemma such that $X = \bigcup_{i \in I} X_i$. The class RX is axiomatized by:

$$\forall x_0, x_1 \left(\bigvee_{i \in I} \left(\bigwedge_{\varphi(x_0, x_1, y_0, \ldots, y_t) \in \Sigma(T_i)} \forall y_0, \ldots, y_t \neg \varphi(x_0, x_1, y_0, \ldots, y_t) \right) \right).$$

A structure A satisfies the above sentence if and only if for any $a, b \in A$ with $a \neq b$ there is some T_i such that $T_i \cup D^+(A) \cup \{\neg(a = b)\}$ is consistent.

By calculating the length of the above conjunctions and disjunction we have:

Corollary. *The class of residually finite groups is axiomatizable by a sentence of $L_{\omega_1\omega}$.*

Using the Löwenheim-Skolem theorem for $L_{\omega_1\omega}$ we can immediately conclude the class of residually finite groups is of countable character. In general we do not use Theorem 2 to show a structure is in RX.

Theorem 3. *Suppose X is as in Theorem 2 and λ is the cardinality of the language of X. If every substructure of A of cardinality λ is in RX then $A \in RX$.*

Proof. For $a, b \in A$ with $a \neq b$, choose $B \prec A$ (B an elementary substructure of A) such that $|B| = \lambda$ and $a, b \in B$. Since $B \in RX_i$ for some i, $B \models \{\forall y_0, \ldots, y_t \neg \varphi(a, b, y_0, \ldots, y_t) | \varphi(x_0, x_1, y_0, \ldots, y_t) \in \Sigma(T_i)\}$ (adopting the notation of Theorem 2). Since $B \prec A$, $T_i \cup D^+(A) \cup \{\neg(a = b)\}$ is consistent.

It is possible to extend these results. Suppose T is a theory such that a homomorphic image of a model of T is itself a model of T (see [**1**, Theorem 3.2.2, p. 126] for a characterization of such theories). We can now relativize to T. Define X to be *universal relative* to T if a model of T which is a substructure of some element of X is itself an element of X. For such X we can derive the above results for models of T being residually in X. This discussion gives the generalization promised in the introduction.

The referee has pointed out a different generalization. The results hold for an X not necessarily universal if SPX is used in place of RX (S denotes taking substructures and P denotes taking Cartesian products).

References

1. C. C. Chang and H. J. Keisler, *Model theory*, North-Holland, Amsterdam, 1973.

2. B. H. Neumann, *Properties of countable character*, Proc. Internat. Congr. Math. (Nice, 1970), vol. 1, Gauthier-Villars, Paris, 1971, pp. 293–296.

3. ______, *Group properties of countable character*, Selected Questions of Algebra and Logic, Izdat. "Nauka" Sibirsk. Otdel., Novosibirsk, 1973, pp. 197–204. MR **48** #6254.

4. G. Sabbagh, *On properties of countable character*, Bull. Austral. Math. Soc. **4** (1971), 183–192.

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