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ON p-TORSION IN ETALE COHOMOLOGY AND IN THE BRAUER GROUP

ROBERT TREGER

ABSTRACT. If X is an affine scheme in characteristic $p > 0$, then $\mathrm{Br}(X)(p) \xrightarrow{\sim} H^2_{\mathrm{et}}(X, \mathbf{G}_m)(p)$ and $H^n_{\mathrm{et}}(X, \mathbf{G}_m)(p) = 0$ for $n \geqslant 3$. This gives a partial answer to the conjecture that the Brauer group of any scheme X is canonically isomorphic to the torsion part of $H^2_{\mathrm{et}}(X, \mathbf{G}_m)$. This result is then applied to prove that $\mathrm{Br}(R)(p)$ is p-divisible where R is a commutative ring of characteristic $p > 0$ (theorem of Knus, Ojanguren and Saltman), and also to construct examples of domains R of characteristic $p > 0$ with large $\mathrm{Ker}(\mathrm{Br}(R)(p) \to \mathrm{Br}(Q)(p))$, where Q is the ring of fractions of R.

The main result of this note (Theorem) is a partial answer to the following well-known

CONJECTURE [4, II, 2]. The Brauer group of any scheme X is canonically isomorphic to the torsion part of the second etale cohomology group of X with coefficients in the sheaf of units $\mathbf{G}_m$, i.e., the image of the inclusion δ: $\mathrm{Br}(X) \to H^2_{\mathrm{et}}(X, \mathbf{G}_m)$ [4, I, Proposition 1.4] coincides with the torsion part of $H^2_{\mathrm{et}}(X, \mathbf{G}_m)$.

The results of this note were obtained in Chicago in the fall of 1976 (cf. [9], [10]). O. Gaber, using a completely different approach,[1] independently proved the following general result: the conjecture is true for $X = U_1 \cup U_2$ where U_1, U_2 are affine schemes. (I hope he will also publish his rather long but very interesting proof.)

1. THEOREM. *Let* $X = \mathrm{Spec}(R)$ *be an affine scheme in characteristic* $p > 0$. *Then* δ: $\mathrm{Br}(X)(p) \xrightarrow{\sim} H^2_{\mathrm{et}}(X, \mathbf{G}_m)(p)$ *and* $H^n_{\mathrm{et}}(X, \mathbf{G}_m)(p) = 0$ *for* $n \geqslant 3$.

PROOF. For any positive integer e we shall consider an extension of R of the form

$$K_e = R[\{x_j | j \in J\}]/(\{x_j^{p^e} - a_j | j \in J\})$$

where $\{a_j | j \in J\}$ is a possibly infinite set of generators for the R^{p^e}-algebra R. Each algebra $K_e = \mathrm{inj\,lim}_{\gamma \in \Gamma}(K_{\gamma,e})$, where

$$K_{\gamma,e} = R[x_{\gamma,1}, \ldots, x_{\gamma,s_\gamma}]/(\{x_{\gamma,i}^{p^e} - a_{\gamma,i} | 1 \leqslant i \leqslant s_\gamma\}), \qquad a_{\gamma,i} \in \{a_j | j \in J\}$$

and $\gamma \in \Gamma$, the index set. All $K_{\gamma,e}$ are free as R-modules.

Let $\mathbf{U}$ be the functor which associates to any commutative R-algebra S its group of units $\mathbf{U}(S)$ [6, V, 1.2]. We denote by $H^n(S/R, \mathbf{U})$ or $H^n(S/R)$ the Amitsur

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[1]The referee pointed out that Gaber's proof is a generalization of Hoobler's proof for smooth affine varieties over a field.

cohomology groups for the functor **U** [**6**, V, 1, 2]. It follows immediately from the definition of the Amitsur complex and Berkson's theorem (see a proof by D. Zelinski in [**6**, V, 5.1]) that

$$H^n(K_e/R) = H^n(K_{\gamma,e}/R) = 0 \quad \text{for } n \geqslant 3, \quad \gamma \in \Gamma. \tag{1}$$

Let us consider a sequence of ring homomorphisms $R \xrightarrow{\varepsilon} K_e \xrightarrow{p^e} R$, where ε is the natural embedding and p^e is the p^e-power map. Let $Y_{\gamma,e} = \operatorname{Spec}(K_{\gamma,e})$ and $Y_e = \operatorname{Spec}(K_e)$. This sequence yields a sequence of homomorphisms of etale cohomology groups:

$$H^n_{\text{et}}(X, \mathbf{G}_m) \xrightarrow{\bar{\varepsilon}} H^n_{\text{et}}(Y_e, \mathbf{G}_m) \xrightarrow{\bar{p}^e} H^n_{\text{et}}(X, \mathbf{G}_m). \tag{2}$$

We shall need two general remarks. If $H^n_{f1}(X, \mathbf{G}_m)$ are the cohomology groups of X in "fppf" topology [**4**, III, 5] then there exist canonical isomorphisms $H^n_{f1}(X, \mathbf{G}_m) \simeq H^n_{\text{et}}(X, \mathbf{G}_m)$ [**4**, III, 11.7]. Moreover, $\bar{p} \cdot \bar{\varepsilon}$ is the p-power map [**6**, V, 1].

Second, for any faithfully flat K_e-algebra B,

$$H^n(B/K_e) = H^n(K_e \otimes_R B/K_e)$$

[**8**, 4.3].

The classical construction of Rosenberg and Zelinsky shows that the natural map $H^2(K_e/R) \to H^2_{\text{et}}(X, \mathbf{G}_m)$ factors through $\operatorname{Br}(K_e/R)$ [**6**, V]. Consider the spectral sequence for the Amitsur complex [**6**, V, 4]. Since the homomorphism η: $K_e^m \to K_e$ given by $\eta(k_1 \otimes \cdots \otimes k_m) = k_1 \cdots k_m$ has a nilpotent kernel for each $m > 1$, by a theorem of Rosenberg and Zelinsky ([**8**, 4.1] or [**6**, V, 4]), the natural sequence

$$H^n(K_e, F_\omega/R) \to H^n(F_\omega/R) \to H^n(K_e \otimes_R F_\omega/K_e) \tag{3}$$

is exact for any etale R-algebra F_ω $(\omega \in \Omega)$. Obviously, $K_e \otimes_R F_\omega$ are etale K_e-algebras. We want to compute $\lim_{\omega \in \Omega} H^n(K_e, F_\omega/R)$. Consider the second spectral sequence with $K = K_e$, $F = F_\omega$. By [**8**, Lemma 3.1], $''E_1^{m,n} = 0$ for all $n > 0$, $m = 0, 1$. Furthermore,

$$''E_1^{3,n} = H^2(K_e \otimes_R F_\omega^n/F_\omega^n)$$

and

$$''E_1^{2,n} = H^1(K_e \otimes_R F_\omega^n/F_\omega^n) = \operatorname{Pic}(K_e \otimes_R F_\omega^n/F_\omega^n).$$

Artin's theorem [**1**] implies that $\lim_{\omega \in \Omega} {}''E_1^{m,n} = 0$ for $m \geqslant 2$, $n > 0$.

Passing to the limit over the directed family F_ω $(\omega \in \Omega)$ in (3) and applying a standard result about spectral sequences [**2**, XV, Theorem 5.12], we get the exact sequence

$$H^n(K_e/R) \xrightarrow{\alpha} H^n_{\text{et}}(X, \mathbf{G}_m) \xrightarrow{\bar{\varepsilon}} H^n_{\text{et}}(Y_e, \mathbf{G}_m). \tag{4}$$

Let $\xi \in {}_{p^e}H^2_{\text{et}}(X, \mathbf{G}_m)$, i.e. ξ has order p^e in the group $H^2_{\text{et}}(X, \mathbf{G}_m)$. Then $\bar{p}^e \cdot \bar{\varepsilon}(\xi) = 0$ (see (2)) hence, by a well-known lemma [**4**, III, 11.8], $\bar{\varepsilon}(\xi) = 0$. Therefore, $\xi \in \operatorname{Im}(\alpha)$. Hence, ξ comes from an Azumaya R-algebra.

If $n \geqslant 3$ then, by (1), $H^n(K_e/R) = 0$. Hence $\xi \in {}_{p^e}H^n_{\text{et}}(X, \mathbf{G}_m)$ implies $\xi = 0$. This proves the theorem.

2. We now give an example of a local ring R of an affine domain over an algebraically closed field with large group $\mathrm{Br}(Q/R)(p) \overset{\mathrm{def}}{=} \mathrm{Ker}(\mathrm{Br}(R)(p) \to \mathrm{Br}(Q)(p))$, where Q is the quotient field of R.

Let k be an algebraically closed field of characteristic $p > 0$. Let $(R, \mathbf{m})$ be a two-dimensional local normal k-domain with the quotient field Q and residue field $R/\mathbf{m} \simeq k$. There is a commutative diagram [**4**, II, 1.7 and (7 bis)]

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Br}(Q/R) & \to & \mathrm{Br}(R) & \to & \mathrm{Br}(Q) \\ & & \tau\downarrow & & \delta\downarrow & & \| \\ 0 & \to & \mathrm{Cl}(R^h)/\mathrm{Cl}(R) & \to & H^2_{\mathrm{et}}(X, \mathbf{G}_m) & \to & \mathrm{Br}(Q) \end{array}$$

where τ is the restriction of δ and R^h is the henselisation of R. It is well known that $\mathrm{Cl}(R^h) = \mathrm{Cl}(\hat{R})$, where $\hat{R}$ is the m-adic completion of R.

Suppose, now that R (as above) is a factorial domain with a nonrational singularity. Then $\mathrm{Cl}(R^h)/\mathrm{Cl}(R) \simeq \mathrm{Cl}(\hat{R})$, and $\mathrm{Cl}(\hat{R})$ contains a nondiscrete commutative subgroup, hence $\mathrm{Cl}(\hat{R})(p) \neq 0$. Thus $\mathrm{Br}(Q/R)(p) \simeq \mathrm{Cl}(\hat{R})(p) \neq 0$.

Example. Let k (as above) be of characteristic 2. Let $A = k[X, Y, Z]/(X^2 + Y^{2i+1} + Z^{2j+1})$ where $(i, j) \neq (1, 1)$, $(1, 2)$, $(2, 1)$ and $(2i + 1, 2j + 1) = 1$. Let $R = A_{\mathbf{m}}$ where $\mathbf{m} = (x, y, z) \subset A$ is an ideal in A generated by the images of X, Y, Z in A. Then R is a factorial domain but $\mathrm{Cl}(\hat{R}) \simeq (k_+)^{r(i,j)}$ where $r(i, j)$ is asymptotic to $ij/2$, by Samuel [**3**, IV, 17]. Thus we can make $\mathrm{Br}(Q/R)(p)$ as large as we wish.

Ojanguren (unpublished) independently constructed examples with nontrivial $\mathrm{Br}(Q/R)$. M. Artin pointed out to me that one can construct examples in characteristic zero with nontrivial $\mathrm{Br}(Q/R)$ by contracting some curves on algebraic $K - 3$ surfaces.

3. Now we present a short functorial proof of the following.

Proposition (Knus-Ojanguren-Saltman; cf. [**7**]). *The Brauer group of any affine scheme X in characteristic $p > 0$ is p-divisible.*

I wish to thank D. Saltman for showing me his proof before it appeared in [**7**].

Proof. Let $X = \mathrm{Spec}(R)$ and K_1, $Y_1 = \mathrm{Spec}(K_1)$ be as in the theorem. There is a standard exact sequence of sheaves on X_{f1} (see, for instance, [**5**, 1.4])

$$0 \to \mathbf{G}_{m,X} \overset{i}{\to} \varphi_* \mathbf{G}_{m,Y_1} \overset{dln}{\to} Z^1_{Y_1/X} \overset{\mathbf{C}-I}{\to} \psi^* \Omega^1_{Y_1/X} \to 0 \tag{5}$$

where φ: $Y_1 \to X$ is the map defined by the inclusion: $R \to K_1$, I is the formal p-power map, ψ: $X \to Y_1$ is the map induced by the map p: $K_1 \to R$, $\mathbf{C}$ is the Cartier operator, and

$$Z^1_{Y_1/X} = \mathrm{Ker}\left[\varphi_* d_{Y_1/X}\colon \varphi_* \Omega^1_{Y_1/X} \to \varphi_* \Omega^2_{Y_1/X}\right]$$

is the sheaf of closed 1-forms. Consider the natural commutative diagram

$$\begin{array}{ccc} H^2_{f1}(X, \mathbf{G}_{m,X}) & \overset{\bar{i}}{\to} & H^2_{f1}(X, \varphi_* \mathbf{G}_{m,Y_1}) \\ F \searrow & & \swarrow W \\ & H^2_{f1}(X, \mathbf{G}_{m,X}) & \end{array}$$

where the map F is induced by the absolute Frobenius on X and W is induced by the map p. Since X is affine and $Z^1_{Y_1/X}$ and $\psi_* \Omega^1_{Y_1/X}$ are quasi-coherent sheaves, $H^2_{fl}(X, \operatorname{Ker}(\mathbf{C} - I)) = 0$. Hence $\bar{i}$ is surjective. It is trivial that W is surjective. Therefore $F = W \cdot i$ is surjective. Since, by the theorem, $\operatorname{Br}(X)(p) \simeq H^2_{\text{et}}(X, \mathbf{G}_m)(p)$, and $H^2_{\text{et}}(X, \mathbf{G}_m)(p) \simeq H^2_{fl}(X, \mathbf{G}_m)(p)$ [**4**, III, 11.7], the Brauer group $\operatorname{Br}(X)$ is p-divisible.

Of course, for general schemes the Brauer group is not p-divisible (cf. [**5**, §2]).

4. Remark. Presumably our method can be applied to the investigation of p-torsion in the nonaffine cases (we used that X in the theorem is affine to conclude that $\operatorname{Ker}(\bar{p}^e) = 0$ in (2)). A straightforward generalization of the theorem to curves can be used to prove an old theorem of M. Artin (unpublished): If f: $V' \to V$ is a proper morphism with fibres of dimension 1 and V' regular of dimension 2, then $R^q f_* \mathbf{G}_{m,V'} = 0$ for $q \geqslant 2$. Indeed, the vanishing of $(R^q f_* \mathbf{G}_{m,V'})(l)$, where l is any prime number, is proved exactly as the analogous result in [**4**, III, 3]; see also [**11**]. The theorem for curves takes care of the case $l = p$, the characteristic of V.

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Institute of Mathematics, Hebrew University, Jerusalem, Israel

Current address: School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540