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A THEOREM OF C. RYLL-NARDZEWSKI AND METRIZABLE L.C.A. GROUPS

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ABSTRACT. Γ denotes a metrizable locally compact abelian group and $\bar{\Gamma}$ its Bohr compactification. Let $\gamma \in \Gamma$ be a cluster point of some subset E of Γ in the topology of $\bar{\Gamma}$. Then there are two disjoint subsets of E which also cluster at γ in the Bohr group topology. The proof is elementary and provides a new proof of the theorem of C. Ryll-Nardzewski on cluster points of I-sets in R. Given the continuum hypothesis, either theorem characterizes metrizability in locally compact abelian groups. One of these characterizations is shown to be equivalent to the continuum hypothesis.

Γ denotes a locally compact abelian group whose dual is G. $\bar{\Gamma}$, the Bohr compactification of Γ, is the l.c.a. group that is dual to G_d, G with the discrete topology.

THEOREM 1. *Suppose that* Γ *is metrizable. Let E be a subset of* Γ *which clusters at* $\gamma \in \Gamma$ *in the topology of* $\bar{\Gamma}$. *Then there are two disjoint subsets of E which likewise cluster at* γ.

PROOF. A basic neighborhood $U = U(\gamma; g_1, \ldots, g_n; \varepsilon)$ of γ in $\bar{\Gamma}$ is determined by a finite set of points $g_1, \ldots, g_n$ from G and some $\varepsilon > 0$. Precisely, U consists of λ in $\bar{\Gamma}$ such that $|\lambda(g_i) - \gamma(g_i)| < \varepsilon$ for $1 \leq i \leq n$. Because γ is a cluster point of E in $\bar{\Gamma}$, given any finite subset $F \subseteq E$ there is some $\lambda \in E \setminus F$ such that $\lambda \in U$. Since both λ and γ are from Γ and thus continuous on G with respect to the original topology, there is some neighborhood V of $(g_1, \ldots, g_n)$ in G^n such that $(h_1, \ldots, h_n) \in V$ implies $\lambda \in U(\gamma; h_1, \ldots, h_n; 3\varepsilon)$. For any compact subset K of G, K^n is covered by a finite number of such V's. We may conclude that for K a compact subset of G, for $n > 0$, for $\varepsilon > 0$, and F a finite subset of E, there exist $\lambda_1, \ldots, \lambda_m$ in $E \setminus F$ such that $(h_1, \ldots, h_n) \in K^n$ implies that for some $1 \leq j \leq m$, $\lambda_j \in U(\gamma; h_1, \ldots, h_n; \varepsilon)$. Denote the set $\{\lambda_1, \ldots, \lambda_m\}$ as $D(K, n, \varepsilon, F)$.

We now use the hypothesis that Γ is metrizable. This occurs exactly when G is σ-compact. Let G be $\bigcup_{n=1}^{\infty} K_n$ with each K_n compact and $K_n \subseteq K_{n+1}$. Define two sequences of finite subsets of E inductively. Set $T_0 = S_0 = \varnothing$. Let $S_{n+1} = D(K_{n+1}, n+1, (n+1)^{-1}, F_{n+1})$ where $F_{n+1} = \bigcup_{j \leq n}(T_j \cup S_j)$. Then $T_{n+1} = D(K_{n+1}, n+1, (n+1)^{-1}, F_{n+1} \cup S_{n+1})$. Clearly $\bigcup_n T_n$ and $\bigcup_n S_n$ are disjoint subsets of E which cluster at γ in $\bar{\Gamma}$. $\square$

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DEFINITION. A subset E of Γ is said to be an *I-set* if every bounded function on E can be extended to $\bar{\Gamma}$ as a continuous function. Equivalently, $E \subset \Gamma$ is an I-set if every bounded function on E is the restriction to E of an almost periodic function on Γ [**1**, p. 32].

The following result due to C. Ryll-Nardzewski [**2**] for $\Gamma = R$ is a corollary of Theorem 1.

COROLLARY. *Let* Γ *be a metrizable l.c.a. group. If* $E \subset \Gamma$ *is an I-set, then E has no points of* Γ *as cluster points in* $\bar{\Gamma}$. *Consequently the union of E with any finite subset of* Γ *is also an I-set.*

The theorem and the corollary are both false when $G = T_d$, the circle with the discrete topology. In that case $\Gamma = \bar{\Gamma} = \bar{Z}$, the Bohr compactification of the integer group Z. The set E which consists of powers of 3 is an I-set, both as a subset of Z and of Γ. The set E has uncountably many cluster points ($\Gamma = \bar{\Gamma}$). This example is characteristic of nonmetrizable groups if the continuum hypothesis is assumed; i.e., $2^{\aleph_0} = \aleph_1$. In what follows (CH) indicates use of the continuum hypothesis.

PROPOSITION (CH). *Let* Γ *be a nondegenerate compact abelian group, and let I be an uncountable set. Then there is an embedding of the Stone-Čech compactification* $\beta(\mathbf{N})$ *in the group* Γ^I, *where* $\mathbf{N}$ *is the set of natural numbers.*

PROOF. Let 2 be the discrete two-point space. Since $\mathbf{N}$ is discrete, the evaluation map ev: $\mathbf{N} \to 2^{2^{\mathbf{N}}}$ extends to an embedding of $\beta(\mathbf{N})$ into $2^{2^{\mathbf{N}}}$. Since I is uncountable, there is an injection of $2^{\mathbf{N}}$ into I, and this then allows an embedding of $2^{2^{\mathbf{N}}}$ into Γ^I, since Γ is nondegenerate and compact. The composition of the embedding of $\beta(\mathbf{N})$ into $2^{2^{\mathbf{N}}}$ with this embedding into Γ^I then yields the desired embedding. □

LEMMA 1. *Let* Γ *and* Γ' *be locally compact abelian groups, and let* $f\colon \Gamma \to \Gamma'$ *be a continuous surmorphism of* Γ *onto* Γ'. *Then any I-set in* Γ' *lifts to an I-set in* Γ.

PROOF. Let E be an I-set in Γ'. For each point of E, choose a point in Γ which maps onto it under f, and call the resulting set F. Then $f|F\colon F \to E$ is a bijection. Since f is a continuous surmorphism, f extends to a continuous surmorphism $\bar{f}$: $\bar{\Gamma} \to \bar{\Gamma}'$.

Now, let $g\colon F \to \mathbf{C}$ be any bounded function. Then $g \cdot f^{-1}$ is a bounded function on the I-set E, and so this extends to a continuous function $g_1\colon \bar{\Gamma}' \to \mathbf{C}$. The composition $g_1 \cdot \bar{f}\colon \bar{\Gamma} \to \mathbf{C}$ then provides the desired extension of g to $\bar{\Gamma}$. Thus F is an I-set in Γ. □

Below ${}^I H$ and H^I denote direct sums and direct products of the group H, respectively.

LEMMA 2. *Let G be an uncountable abelian group. Then G has a subgroup G′ which is isomorphic to* ${}^I H$, *where H is a nondegenerate group, and I is an uncountable set* (*of the same cardinality as G*).

PROOF. Let D be the divisible hull of G, i.e., D is aminimal divisible extension of G. Then D can be represented as ${}^{(r_0)}\mathbf{Q} \oplus \bigoplus_p^{(r_p)} Z(p^\infty)$ where $\mathbf{Q}$ is the group of rationals, r_0 is the torsion-free rank of G, and r_p is the p-rank of G for each prime p in P, the set of primes [3, Appendix A, pp. 444–446]. Now, D is uncountable since G is (in fact of equal cardinality), and so either r_0 is uncountable, or r_p is uncountable for some prime p. In any event, we conclude that G has an uncountable set of independent elements all having the same order. The group G' that this set generates is then isomorphic to ${}^I H$, where H is the cyclic group of the order in question, and I is the set of independent elements of the order. □

Theorem 2 below is a converse of Theorem 1 under the continuum hypothesis. Theorems 1 and 2 together characterize metrizable locally compact abelian groups under the continuum hypothesis assumption.

THEOREM 2 (CH). *Let* Γ *be a nonmetrizable l.c.a. group. Then there is an infinite* I*-set* $E \subset \Gamma$ *whose closure in* $\bar{\Gamma}$ *is a subset of* Γ. *Consequently, Theorem* 1 *and its corollary are false for* Γ.

PROOF. Suppose that Γ is a nonmetrizable locally compact abelian group. Then, according to the Principle Structure Theorem for locally compact abelian groups [1, p. 40], Γ has an open subgroup which is a direct product of a vector group V and a compact subgroup Γ' of Γ. Since V is metrizable and Γ is not, and since $V \times \Gamma'$ is open, it follows that Γ' is also not metrizable. Now, Tietze's Extension Theorem implies that any I-set in Γ' is also an I-set in Γ. Moreover, since Γ' is compact and the map from Γ to $\bar{\Gamma}$ is continuous, the image of Γ' in $\bar{\Gamma}$ is closed, and so any I-set in Γ' has its closure a subset of Γ', and hence of the image of Γ in $\bar{\Gamma}$. This shows that it suffices to produce the desired I-set in Γ', or said another way, it suffices to consider the case when Γ is compact.

Given that Γ is a compact nonmetrizable abelian group, its character group G is then an uncountable abelian group. Thus, Lemma 2 implies that G has a subgroup $G' \simeq^I H$, where H is a nondegenerate group, and I is uncountable. Then, the character group Γ' of G' is a quotient of Γ, and is isomorphic to Δ^I, where Δ is the character group of H. Lemma 1 implies that it suffices to produce the desired I-set in Γ'. Moreover, the proposition implies that $\beta(\mathbf{N})$ is embedded in Γ', and this clearly makes the corresponding copy of $\mathbf{N}$ an I-set in Γ', and we are done. □

The existence of an infinite I-set in an arbitrary nonmetrizable l.c.a. group is actually equivalent to the continuum hypothesis. To see this, let G be a discrete group whose cardinality is the first uncountable cardinal and let E be an infinite I-set in $G^\wedge = \Gamma$. For each partition of the set E into two disjoint subsets, choose a finitely-supported discrete measure with rational coefficients whose Fourier-Stieltjes transform is approximately 1 on one subset of the partition of E and approximately 0 on the other subset of E. The cardinality of the set of such discrete measures is the same as that of G but the cardinality of possible partitions of E is at least $2^{\aleph_0}$. Thus, $2^{\aleph_0} \leq \aleph_1$, i.e. the continuum hypothesis holds.

It is also false (without CH) that the theorem of C. Ryll-Nardzewski is characteristic of metrizable l.c.a. groups. This follows from the preceding observation.

Otherwise we could have a compact, nonmetrizable group in which every I-set were finite and hence the Ryll-Nardzewski theorem would hold for that nonmetrizable group.

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