PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, February 1980

l^∞/c_0 HAS NO EQUIVALENT STRICTLY CONVEX NORM

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ABSTRACT. It is shown that the quotient space l^∞/c_0 does not admit an equivalent strictly convex norm.

Introduction. We say that a normed space X, $\|\ \|$ is strictly convex provided $\|x + y\| < 2$ whenever $\|x\| = \|y\| = 1$ and $x \neq y$. This means also that any member x^* in the dual X^* of X ($x^* \neq 0$) attains its norm in at most one point of the unit ball of X. A space is called strictly convexifiable if there exists an equivalent strictly convex norm on the space.

It is known that $l^\infty = l^\infty(\mathbf{N})$ is strictly convexifiable (cf. [1], [2]). However the purpose of this note is to prove that the quotient l^∞/c_0 fails this property, a problem raised independently by J. J. Schäffer and J. Diestel [3].

We will denote infinite subsets of the integers $\mathbf{N}$ by letters L, M, N, P. If L is an infinite subset of $\mathbf{N}$, let $P_\infty(L)$ be the set of all infinite subsets of L. Our proof is based on the following elementary result.

LEMMA. *Suppose $x^* \in (l^\infty)^*$, $L \in P_\infty(\mathbf{N})$ and $\epsilon > 0$. Then there is some $M \in P_\infty(L)$ such that $|x^*(x)| < \epsilon$ whenever $x \in l^\infty$, $\|x\| = 1$ and $x_n = 0$ if $n \notin M$.*

PROOF. We may, of course, assume $\|x^*\| = 1$. Take an integer $d > \epsilon^{-1}$ and disjoint members M_i ($1 \leqslant i \leqslant d$) of $P_\infty(L)$.

If each M_i fails the property, then we find elements $x^{(i)}$ ($1 \leqslant i \leqslant d$) in l^∞, so that:

(1) $\|x^{(i)}\| = 1$,

(2) $x^{(i)}_m = 0$ if $n \notin M_i$, and

(3) $x^*(x^{(i)}) \geqslant \epsilon$.

Consider now the vector $x = x^{(1)} + x^{(2)} + \cdots + x^{(d)}$. Obviously $\|x\| = 1$ and $x^*(x) \geqslant d$. This is the required contradiction.

We are now ready to prove the following theorem.

THEOREM. *Let $|||\ |||$ be an equivalent norm on l^∞/c_0. Then $|||\ |||$ is not strictly convex.*

PROOF. Let $Y = l^\infty/c_0$, $|||\ |||$ and π: $l^\infty \to Y$ the quotient map be given. Let (ϵ_n) be a sequence of positive numbers converging to 0. We make the following construction: Take $F_1 = \{x \in l^\infty;\ \|x\| \leqslant 1\}$ and $s_1 = \sup\{|||\pi(x)|||;\ x \in F_1\}$. Let $x^{(1)} \in F_1$ be such that $|||\pi(x^{(1)})||| > s_1 - \epsilon_1$ and

Received by the editors December 21, 1978.

AMS (MOS) subject classifications (1970). Primary 46B05.

consider $y_1^* \in Y^*$, $|||y_1^*||| = 1$ with $y_1^* \pi(x^{(1)}) > s_1 - \epsilon_1$.

Since $\pi^*(y_1^*) \in (l^\infty)^*$, application of the lemma provides $L_1 \in P_\infty(\mathbf{N})$ such that $|y_1^* \pi(x)| < \epsilon_1$ if $x \in l^\infty$, $x \leqslant 1$ and $x_n = 0$ for $x \notin L_1$. Take $F_2 = \{x \in F_1;\, x_n = x_n^{(1)}$ for $n \notin L_1\}$ and $s_2 = \sup\{|||\pi(x)|||;\, x \in F_2\}$. Let $x^{(2)} \in F_2$ be such that $|||\pi(x^{(2)})||| > s_2 - \epsilon_2$ and consider $y_2^* \in Y^*$, $|||y_2^*||| = 1$ with $y_2^* \pi(x^{(2)}) > s_2 - \epsilon_2$. Again by the lemma, we get $L_2 \in P_\infty(L_1)$ such that $|y_2^* \pi(x)| < \epsilon_2$ for $x \in l^\infty$, $x \leqslant 1$ and $x_n = 0$ if $n \notin L_2$.

In general $F_{i+1} = \{x \in F_i;\, x_n = x_n^{(i)}$ for $n \notin L_i\}$ and $s_{i+1} = \sup\{|||\pi(x)|||;\, x \in F_{i+1}\}$. Let $x^{(i+1)} \in F_{i+1}$ satisfy $|||\pi(x^{(i+1)})||| > s_{i+1} - \epsilon_{i+1}$ and take $y_{i+1}^* \in Y^*$, $|||y_{i+1}^*||| = 1$ with $y_{i+1}^* \pi(x^{(i+1)}) > s_{i+1} - \epsilon_{i+1}$. By the lemma, there is some $L_{i+1} \in P_\infty(L_i)$ so that $|y_{i+1}^* \pi(x)| < \epsilon_{i+1}$ if $x \in l^\infty$, $x \leqslant 1$ and $x_n = 0$ for $n \notin L_{i+1}$.

Since for $x \in F_{i+1}$ we have $\|x - x^{(i)}\| \leqslant 2$ and $x_n - x_n^{(i)} = 0$ for $n \notin L_i$, it follows that $|y_i^* \pi(x - x^{(i)})| < 2\epsilon_i$ and thus $y_i^* \pi(x) > y_i^* \pi(x^{(i)}) - 2\epsilon_i > s_i - 3\epsilon_i$.

Clearly (F_i) is a decreasing sequence of nonvoid $\sigma(l^\infty, l^1)$ compact subsets of l^∞. Also (s_i) decreases and we let $s = \lim_{i \to \infty} s_i$. Take some element x^∞ in $\bigcap_i F_i$ and let y^* be a ω^*-cluster point of (y_i^*). We consider the subset $S = \bigcap_i \pi(F_i)$ of Y. For a fixed $y \in S$, we find $|||y||| \leqslant s$.

Since $y \in \pi(F_{i+1})$, it follows that $y_i^*(y) > s_i - 3\epsilon_i \geqslant s - 3\epsilon_i$ for all i. Consequently $y^*(y) \geqslant s$ and thus $y^*(y) = s = |||y|||$. We show that S contains more than one point. In particular, this will imply that $s > 0$ and $|||y^*||| = 1$. So the proof will be complete. Because (L_i) is decreasing in $P_\infty(\mathbf{N})$, there is some $L \in P_\infty(\mathbf{N})$ with $L \setminus L_i$ finite for all i. Assume $x \in l^\infty$, $\|x\| = 1$ and $x_n = x_n^\infty$ for $n \notin L$. It is possible to take $\pi(x) \neq \pi(x^\infty)$ since L is infinite. We claim that $\pi(x) \in S$. To see this, fix some i and remark that there is $x' \in l^\infty$, $\|x'\| = 1$, $x'_n = x_n$ for $n \notin L \setminus L_i$ and $x'_n = x_n^\infty$ for $n \notin L_i$ (x' depends of course on i). Thus $x'_n = x_n^\infty = x_n^{(j)}$ if $n \notin L_j$, for all $j = 1, \ldots, i - 1$. Now $x' \in F_1$ and proceeding by induction we see that $x' \in F_j$ $(1 \leqslant j \leqslant i)$. Because $L \setminus L_i$ is finite, $\pi(x) = \pi(x') \in \pi(F_i)$. Consequently $\pi(x) \in \bigcap_i \pi(F_i)$, which is what must be obtained.

References

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