PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, February 1980

# $l^\infty/c_0$ HAS NO EQUIVALENT STRICTLY CONVEX NORM

J. BOURGAIN

ABSTRACT. It is shown that the quotient space $l^\infty/c_0$ does not admit an equivalent strictly convex norm.

**Introduction.** We say that a normed space $X$, $\| \ \|$ is strictly convex provided $\|x + y\| < 2$ whenever $\|x\| = \|y\| = 1$ and $x \neq y$. This means also that any member $x^*$ in the dual $X^*$ of $X$ ($x^* \neq 0$) attains its norm in at most one point of the unit ball of $X$. A space is called strictly convexifiable if there exists an equivalent strictly convex norm on the space.

It is known that $l^\infty = l^\infty(\mathbf{N})$ is strictly convexifiable (cf. [1], [2]). However the purpose of this note is to prove that the quotient $l^\infty/c_0$ fails this property, a problem raised independently by J. J. Schäffer and J. Diestel [3].

We will denote infinite subsets of the integers $\mathbf{N}$ by letters $L$, $M$, $N$, $P$. If $L$ is an infinite subset of $\mathbf{N}$, let $P_\infty(L)$ be the set of all infinite subsets of $L$. Our proof is based on the following elementary result.

LEMMA. *Suppose $x^* \in (l^\infty)^*$, $L \in P_\infty(\mathbf{N})$ and $\epsilon > 0$. Then there is some $M \in P_\infty(L)$ such that $|x^*(x)| < \epsilon$ whenever $x \in l^\infty$, $\|x\| = 1$ and $x_n = 0$ if $n \notin M$.*

PROOF. We may, of course, assume $\|x^*\| = 1$. Take an integer $d > \epsilon^{-1}$ and disjoint members $M_i$ ($1 \leqslant i \leqslant d$) of $P_\infty(L)$.

If each $M_i$ fails the property, then we find elements $x^{(i)}$ ($1 \leqslant i \leqslant d$) in $l^\infty$, so that:

(1) $\|x^{(i)}\| = 1$,

(2) $x^{(i)}_m = 0$ if $n \notin M_i$, and

(3) $x^*(x^{(i)}) \geqslant \epsilon$.

Consider now the vector $x = x^{(1)} + x^{(2)} + \cdots + x^{(d)}$. Obviously $\|x\| = 1$ and $x^*(x) \geqslant d$. This is the required contradiction.

We are now ready to prove the following theorem.

THEOREM. *Let $||| \ |||$ be an equivalent norm on $l^\infty/c_0$. Then $||| \ |||$ is not strictly convex.*

PROOF. Let $Y = l^\infty/c_0$, $||| \ |||$ and $\pi\colon l^\infty \to Y$ the quotient map be given. Let $(\epsilon_n)$ be a sequence of positive numbers converging to 0. We make the following construction: Take $F_1 = \{x \in l^\infty; \|x\| \leqslant 1\}$ and $s_1 = \sup\{|||\pi(x)|||; x \in F_1\}$. Let $x^{(1)} \in F_1$ be such that $|||\pi(x^{(1)})||| > s_1 - \epsilon_1$ and

Received by the editors December 21, 1978.

*AMS (MOS) subject classifications* (1970). Primary 46B05.


0002-9939/80/0000-0065/$01.50

consider $y_1^* \in Y^*$, $|||y_1^*||| = 1$ with $y_1^*\pi(x^{(1)}) > s_1 - \epsilon_1$.

Since $\pi^*(y_1^*) \in (l^\infty)^*$, application of the lemma provides $L_1 \in P_\infty(\mathbf{N})$ such that $|y_1^*\pi(x)| < \epsilon_1$ if $x \in l^\infty$, $x \leqslant 1$ and $x_n = 0$ for $x \notin L_1$. Take $F_2 = \{x \in F_1;\ x_n = x_n^{(1)}$ for $n \notin L_1\}$ and $s_2 = \sup\{|||\pi(x)|||;\ x \in F_2\}$. Let $x^{(2)} \in F_2$ be such that $|||\pi(x^{(2)})||| > s_2 - \epsilon_2$ and consider $y_2^* \in Y^*$, $|||y_2^*||| = 1$ with $y_2^*\pi(x^{(2)}) > s_2 - \epsilon_2$. Again by the lemma, we get $L_2 \in P_\infty(L_1)$ such that $|y_2^*\pi(x)| < \epsilon_2$ for $x \in l^\infty$, $x \leqslant 1$ and $x_n = 0$ if $n \notin L_2$.

In general $F_{i+1} = \{x \in F_i;\ x_n = x_n^{(i)}$ for $n \notin L_i\}$ and $s_{i+1} = \sup\{|||\pi(x)|||;\ x \in F_{i+1}\}$. Let $x^{(i+1)} \in F_{i+1}$ satisfy $|||\pi(x^{(i+1)})||| > s_{i+1} - \epsilon_{i+1}$ and take $y_{i+1}^* \in Y^*$, $|||y_{i+1}^*||| = 1$ with $y_{i+1}^*\pi(x^{(i+1)}) > s_{i+1} - \epsilon_{i+1}$. By the lemma, there is some $L_{i+1} \in P_\infty(L_i)$ so that $|y_{i+1}^*\pi(x)| < \epsilon_{i+1}$ if $x \in l^\infty$, $x \leqslant 1$ and $x_n = 0$ for $n \notin L_{i+1}$.

Since for $x \in F_{i+1}$ we have $\|x - x^{(i)}\| \leqslant 2$ and $x_n - x_n^{(i)} = 0$ for $n \notin L_i$, it follows that $|y_i^*\pi(x - x^{(i)})| < 2\epsilon_i$ and thus $y_i^*\pi(x) > y_i^*\pi(x^{(i)}) - 2\epsilon_i > s_i - 3\epsilon_i$.

Clearly $(F_i)$ is a decreasing sequence of nonvoid $\sigma(l^\infty, l^1)$ compact subsets of $l^\infty$. Also $(s_i)$ decreases and we let $s = \lim_{i\to\infty} s_i$. Take some element $x^\infty$ in $\bigcap_i F_i$ and let $y^*$ be a $\omega^*$-cluster point of $(y_i^*)$. We consider the subset $S = \bigcap_i \pi(F_i)$ of $Y$. For a fixed $y \in S$, we find $|||y||| \leqslant s$.

Since $y \in \pi(F_{i+1})$, it follows that $y_i^*(y) > s_i - 3\epsilon_i \geqslant s - 3\epsilon_i$ for all $i$. Consequently $y^*(y) \geqslant s$ and thus $y^*(y) = s = |||y|||$. We show that $S$ contains more than one point. In particular, this will imply that $s > 0$ and $|||y^*||| = 1$. So the proof will be complete. Because $(L_i)$ is decreasing in $P_\infty(\mathbf{N})$, there is some $L \in P_\infty(\mathbf{N})$ with $L \setminus L_i$ finite for all $i$. Assume $x \in l^\infty$, $\|x\| = 1$ and $x_n = x_n^\infty$ for $n \notin L$. It is possible to take $\pi(x) \neq \pi(x^\infty)$ since $L$ is infinite. We claim that $\pi(x) \in S$. To see this, fix some $i$ and remark that there is $x' \in l^\infty$, $\|x'\| = 1$, $x'_n = x_n$ for $n \notin L \setminus L_i$ and $x'_n = x_n^\infty$ for $n \notin L_i$ ($x'$ depends of course on $i$). Thus $x'_n = x_n^\infty = x_n^{(j)}$ if $n \notin L_j$, for all $j = 1, \ldots, i - 1$. Now $x' \in F_1$ and proceeding by induction we see that $x' \in F_j$ $(1 \leqslant j \leqslant i)$. Because $L \setminus L_i$ is finite, $\pi(x) = \pi(x') \in \pi(F_i)$. Consequently $\pi(x) \in \bigcap_i \pi(F_i)$, which is what must be obtained.

## References

1. M. M. Day, *Normed linear spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 21, Springer-Verlag, New York-Heidelberg, 1973.

2. J. Diestel, *Geometry of Banach spaces.Selected topics*, Lecture Notes in Math., vol. 485, Springer-Verlag, Berlin and New York, 1976.

3. ______, Private communication.

Department of Mathematics, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussels, Belgium