

## $l^\infty/c_0$ HAS NO EQUIVALENT STRICTLY CONVEX NORM

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**ABSTRACT.** It is shown that the quotient space  $l^\infty/c_0$  does not admit an equivalent strictly convex norm.

**Introduction.** We say that a normed space  $X$ ,  $\| \cdot \|$  is strictly convex provided  $\|x + y\| < 2$  whenever  $\|x\| = \|y\| = 1$  and  $x \neq y$ . This means also that any member  $x^*$  in the dual  $X^*$  of  $X$  ( $x^* \neq 0$ ) attains its norm in at most one point of the unit ball of  $X$ . A space is called strictly convexifiable if there exists an equivalent strictly convex norm on the space.

It is known that  $l^\infty = l^\infty(\mathbb{N})$  is strictly convexifiable (cf. [1], [2]). However the purpose of this note is to prove that the quotient  $l^\infty/c_0$  fails this property, a problem raised independently by J. J. Schäffer and J. Diestel [3].

We will denote infinite subsets of the integers  $\mathbb{N}$  by letters  $L, M, N, P$ . If  $L$  is an infinite subset of  $\mathbb{N}$ , let  $P_\infty(L)$  be the set of all infinite subsets of  $L$ . Our proof is based on the following elementary result.

**LEMMA.** Suppose  $x^* \in (l^\infty)^*$ ,  $L \in P_\infty(\mathbb{N})$  and  $\epsilon > 0$ . Then there is some  $M \in P_\infty(L)$  such that  $|x^*(x)| < \epsilon$  whenever  $x \in l^\infty$ ,  $\|x\| = 1$  and  $x_n = 0$  if  $n \notin M$ .

**PROOF.** We may, of course, assume  $\|x^*\| = 1$ . Take an integer  $d > \epsilon^{-1}$  and disjoint members  $M_i$  ( $1 \leq i \leq d$ ) of  $P_\infty(L)$ .

If each  $M_i$  fails the property, then we find elements  $x^{(i)}$  ( $1 \leq i \leq d$ ) in  $l^\infty$ , so that:

- (1)  $\|x^{(i)}\| = 1$ ,
- (2)  $x_n^{(i)} = 0$  if  $n \notin M_i$ , and
- (3)  $x^*(x^{(i)}) > \epsilon$ .

Consider now the vector  $x = x^{(1)} + x^{(2)} + \dots + x^{(d)}$ . Obviously  $\|x\| = 1$  and  $x^*(x) > d$ . This is the required contradiction.

We are now ready to prove the following theorem.

**THEOREM.** Let  $\| \cdot \|$  be an equivalent norm on  $l^\infty/c_0$ . Then  $\| \cdot \|$  is not strictly convex.

**PROOF.** Let  $Y = l^\infty/c_0$ ,  $\| \cdot \|$  and  $\pi: l^\infty \rightarrow Y$  the quotient map be given. Let  $(\epsilon_n)$  be a sequence of positive numbers converging to 0. We make the following construction: Take  $F_1 = \{x \in l^\infty; \|x\| \leq 1\}$  and  $s_1 = \sup\{\|\pi(x)\|; x \in F_1\}$ . Let  $x^{(1)} \in F_1$  be such that  $\|\pi(x^{(1)})\| > s_1 - \epsilon_1$  and

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consider  $y_1^* \in Y^*$ ,  $\|y_1^*\| = 1$  with  $y_1^* \pi(x^{(1)}) > s_1 - \epsilon_1$ .

Since  $\pi^*(y_1^*) \in (l^\infty)^*$ , application of the lemma provides  $L_1 \in P_\infty(\mathbb{N})$  such that  $|y_1^* \pi(x)| < \epsilon_1$  if  $x \in l^\infty$ ,  $x \leq 1$  and  $x_n = 0$  for  $x \notin L_1$ . Take  $F_2 = \{x \in F_1; x_n = x_n^{(1)} \text{ for } n \notin L_1\}$  and  $s_2 = \sup\{\|\pi(x)\|; x \in F_2\}$ . Let  $x^{(2)} \in F_2$  be such that  $\|\pi(x^{(2)})\| > s_2 - \epsilon_2$  and consider  $y_2^* \in Y^*$ ,  $\|y_2^*\| = 1$  with  $y_2^* \pi(x^{(2)}) > s_2 - \epsilon_2$ . Again by the lemma, we get  $L_2 \in P_\infty(L_1)$  such that  $|y_2^* \pi(x)| < \epsilon_2$  for  $x \in l^\infty$ ,  $x \leq 1$  and  $x_n = 0$  if  $n \notin L_2$ .

In general  $F_{i+1} = \{x \in F_i; x_n = x_n^{(i)} \text{ for } n \notin L_i\}$  and  $s_{i+1} = \sup\{\|\pi(x)\|; x \in F_{i+1}\}$ . Let  $x^{(i+1)} \in F_{i+1}$  satisfy  $\|\pi(x^{(i+1)})\| > s_{i+1} - \epsilon_{i+1}$  and take  $y_{i+1}^* \in Y^*$ ,  $\|y_{i+1}^*\| = 1$  with  $y_{i+1}^* \pi(x^{(i+1)}) > s_{i+1} - \epsilon_{i+1}$ . By the lemma, there is some  $L_{i+1} \in P_\infty(L_i)$  so that  $|y_{i+1}^* \pi(x)| < \epsilon_{i+1}$  if  $x \in l^\infty$ ,  $x \leq 1$  and  $x_n = 0$  for  $n \notin L_{i+1}$ .

Since for  $x \in F_{i+1}$  we have  $\|x - x^{(i)}\| \leq 2$  and  $x_n - x_n^{(i)} = 0$  for  $n \notin L_i$ , it follows that  $|y_i^* \pi(x - x^{(i)})| < 2\epsilon_i$  and thus  $y_i^* \pi(x) > y_i^* \pi(x^{(i)}) - 2\epsilon_i > s_i - 3\epsilon_i$ .

Clearly  $(F_i)$  is a decreasing sequence of nonvoid  $\sigma(l^\infty, l^1)$  compact subsets of  $l^\infty$ . Also  $(s_i)$  decreases and we let  $s = \lim_{i \rightarrow \infty} s_i$ . Take some element  $x^\infty$  in  $\bigcap_i F_i$  and let  $y^*$  be a  $\omega^*$ -cluster point of  $(y_i^*)$ . We consider the subset  $S = \bigcap_i \pi(F_i)$  of  $Y$ . For a fixed  $y \in S$ , we find  $\|y\| \leq s$ .

Since  $y \in \pi(F_{i+1})$ , it follows that  $y_i^*(y) > s_i - 3\epsilon_i \geq s - 3\epsilon_i$  for all  $i$ . Consequently  $y^*(y) \geq s$  and thus  $y^*(y) = s = \|y\|$ . We show that  $S$  contains more than one point. In particular, this will imply that  $s > 0$  and  $\|y^*\| = 1$ . So the proof will be complete. Because  $(L_i)$  is decreasing in  $P_\infty(\mathbb{N})$ , there is some  $L \in P_\infty(\mathbb{N})$  with  $L \setminus L_i$  finite for all  $i$ . Assume  $x \in l^\infty$ ,  $\|x\| = 1$  and  $x_n = x_n^\infty$  for  $n \notin L$ . It is possible to take  $\pi(x) \neq \pi(x^\infty)$  since  $L$  is infinite. We claim that  $\pi(x) \in S$ . To see this, fix some  $i$  and remark that there is  $x' \in l^\infty$ ,  $\|x'\| = 1$ ,  $x'_n = x_n$  for  $n \notin L \setminus L_i$  and  $x'_n = x_n^\infty$  for  $n \notin L_i$  ( $x'$  depends of course on  $i$ ). Thus  $x'_n = x_n^\infty = x_n^{(j)}$  if  $n \notin L_j$ , for all  $j = 1, \dots, i - 1$ . Now  $x' \in F_1$  and proceeding by induction we see that  $x' \in F_j$  ( $1 < j < i$ ). Because  $L \setminus L_i$  is finite,  $\pi(x) = \pi(x') \in \pi(F_i)$ . Consequently  $\pi(x) \in \bigcap_i \pi(F_i)$ , which is what must be obtained.

## REFERENCES

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