PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, February 1980

ON L^1 ISOMORPHISMS

MICHAEL CAMBERN

ABSTRACT. Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be two σ-finite measure spaces. We show that any isomorphism T of the Banach space $L^1(X_1, \Sigma_1, \mu_1)$ onto the Banach space $L^1(X_2, \Sigma_2, \mu_2)$ which satisfies $\|T\| \, \|T^{-1}\| < 2$ induces a transformation of the underlying measure spaces.

In [1] and [2] it has been shown by D. Amir and M. Cambern that if Y_1 and Y_2 are compact Hausdorff spaces, and if there exists an isomorphism T of $C(Y_1)$ onto $C(Y_2)$ with $\|T\| \, \|T^{-1}\| < 2$, then Y_1 and Y_2 are homeomorphic. In this note, we use this theorem to prove an analogous result for L^1 spaces. (Concerning the terminology "regular set isomorphism" as it is used in this paper, the reader is referred to [6].)

THEOREM. *Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be σ-finite measure spaces. If there exists an isomorphism T of $L^1(X_1, \Sigma_1, \mu_1)$ onto $L^1(X_2, \Sigma_2, \mu_2)$ satisfying $\|T\| \, \|T^{-1}\| < 2$, then there exists a regular set isomorphism Φ of (X_1, Σ_1, μ_1) onto (X_2, Σ_2, μ_2).*

PROOF. Since the measure spaces are σ-finite, the dual space of $L^1(X_i, \Sigma_i, \mu_i)$ is $L^\infty(X_i, \Sigma_i, \mu_i)$, $i = 1, 2$ [4, p. 289]. Hence the adjoint transformation T^* is an isomorphism of $L^\infty(X_2, \Sigma_2, \mu_2)$ onto $L^\infty(X_1, \Sigma_1, \mu_1)$ satisfying $\|T^*\| \, \|T^{*-1}\| < 2$. Now $L^\infty(X_i, \Sigma_i, \mu_i)$ is isometrically isomorphic to $C(Y_i)$, $i = 1, 2$, under the map $\rho_i(f) = \hat{f}$, where Y_i is the maximal ideal space of $L^\infty(X_i, \Sigma_i, \mu_i)$ and ρ_i is the Gelfand representation of $L^\infty(X_i, \Sigma_i, \mu_i)$, ([4, p. 445] or [5, p. 17]). Define a map R of $C(Y_2)$ to $C(Y_1)$ by $R(\hat{f}) = \rho_1 \circ T^* \circ \rho_2^{-1}(\hat{f})$, for $\hat{f} \in C(Y_2)$. Then clearly R is an isomorphism of $C(Y_2)$ onto $C(Y_1)$ with $\|R\| \, \|R^{-1}\| < 2$.

It thus follows that there exists a homeomorphism τ mapping Y_1 onto Y_2. And, being a homeomorphism, τ carries the clopen sets of Y_1 onto the clopen sets of Y_2. Now if $A_i \in \Sigma_i$, then $\hat{\chi}_{A_i}$ is the characteristic function of a clopen subset U_{A_i} of Y_i, and every clopen subset U of Y_i is of the form U_{A_i}, for some $A_i \in \Sigma_i$, $i = 1, 2$ [5, p. 17]. Let Φ be the map from Σ_1 to Σ_2, defined modulo null sets by $\Phi(A_1) = A_2$ if $\tau(U_{A_1}) = U_{A_2}$, where $U_{A_i} \subseteq Y_i$ and is related to $A_i \in \Sigma_i$ as in the previous sentence.

If $\mathfrak{N}_i$ denotes the family of null sets in Σ_i, then for $i = 1, 2$, $\Sigma_i / \mathfrak{N}_i$ is isomorphic as a Boolean algebra with the clopen subsets of Y_i, under the correspondence $A_i \leftrightarrow U_{A_i}$. Moreover, the Boolean supremum of a sequence U_{A_i} of clopen sets is the topological closure of the point set union of the U_{A_i}. Then, since the homeomorphism τ of Y_1 onto Y_2 preserves both point set unions and topological closures, it clearly effects an order isomorphism between the Boolean algebras of clopen sets

Received by the editors December 7, 1978 and, in revised form, March 13, 1979.
AMS (MOS) subject classifications (1970). Primary 46E30, 46E15.

of the Y_i, from which it readily follows that the map Φ of the previous paragraph is a regular set isomorphism.

REMARKS AND PROBLEMS. (a) The condition $\|T\|\,\|T^{-1}\| < 2$ in our theorem cannot be removed to allow for arbitrary isomorphisms of $L^1(X_1, \Sigma_1, \mu_1)$ onto $L^1(X_2, \Sigma_2, \mu_2)$ as the following example shows. Let (X_1, Σ_1, μ_1) be the measure space where $X_1 = [0, 1]$, Σ_1 is the σ-field of Lebesgue subsets of $[0, 1]$ and μ_1 is Lebesgue measure. Let (X_2, Σ_2, μ_2) be defined as follows: $X_2 = [0, 1] \cup \{2\}$, Σ_2 consists of the Lebesgue measurable subsets of X_2, and μ_2 is the sum of Lebesgue measure on Σ_2 and of the unit point mass concentrated at 2. For each $k = 0, 1, 2, \ldots$, let I_k be the subset of $[0, 1]$ defined by $I_k = [(2^k - 1)/2^k, (2^{k+1} - 1)/2^{k+1})$. We define a map T from $L^1(X_1, \Sigma_1, \mu_1)$ to $L^1(X_2, \Sigma_2, \mu_2)$ by

$$(T(f))(2) = \int_{I_0} f(t)\,dt$$

and

$$(T(f))(t) = f(t) - 2^{k+1}\int_{I_k} f(t)\,dt + 2^{k+1}\int_{I_{k+1}} f(t)\,dt$$

for $f \in L^1(X_1, \Sigma_1, \mu_1)$ and $t \in I_k$. ($T(f)$ has not been defined at 1, but since we are actually defining a map of equivalence classes rather than functions, the value of $(T(f))(1)$ is of no concern.) It is clear that T is linear. It is moreover one-one and surjective since given $g \in L^1(X_2, \Sigma_2, \mu_2)$, the element $f \in L^1(X_1, \Sigma_1, \mu_1)$ defined by

$$f(t) = g(t) - 2\int_{I_0} g(t)\,dt + 2 \cdot g(2)$$

for $t \in I_0$, and

$$f(t) = g(t) - 2^{k+1}\int_{I_k} g(t)\,dt + 2^{k+1}\int_{I_{k-1}} g(t)\,dt$$

for $t \in I_k$, $k > 0$, is such that $T(f) = g$. Thus T is a continuous isomorphism of $L^1(X_1, \Sigma_1, \mu_1)$ onto $L^1(X_2, \Sigma_2, \mu_2)$. However, since (X_2, Σ_2, μ_2) contains an atom while (X_1, Σ_1, μ_1) does not, there can exist no regular set isomorphism of (X_1, Σ_1, μ_1) onto (X_2, Σ_2, μ_2).

(b) It is known (see [3]) that for the theorem mentioned in the first paragraph of this article, 2 can be replaced by no larger number in the condition $\|T\|\,\|T^{-1}\| < 2$. Is 2 also the "best" number for a theorem of the type obtained in this paper?

(c) Can a theorem analogous to the one of this article be established for L^p, $1 < p < \infty$, $p \neq 2$?

REFERENCES

1. D. Amir, *On isomorphisms of continuous function spaces*, Israel J. Math. **3** (1965), 205–210.

2. M. Cambern, *On isomorphisms with small bound*, Proc. Amer. Math. Soc. **18** (1967), 1062–1066.

3. H. B. Cohen, *A bound-two isomorphism between $C(X)$ Banach spaces*, Proc. Amer. Math. Soc. **50** (1975), 215–217.

4. N. Dunford and J. T. Schwartz, *Linear operators*. I, Interscience, New York, 1958.

5. T. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N.J., 1969.

6. J. Lamperti, *On the isometries of certain function spaces*, Pacific J. Math. **8** (1958), 459–466.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106