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ON L^1 ISOMORPHISMS

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ABSTRACT. Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be two σ-finite measure spaces. We show that any isomorphism T of the Banach space $L^1(X_1, \Sigma_1, \mu_1)$ onto the Banach space $L^1(X_2, \Sigma_2, \mu_2)$ which satisfies $\|T\|\,\|T^{-1}\| < 2$ induces a transformation of the underlying measure spaces.

In [1] and [2] it has been shown by D. Amir and M. Cambern that if Y_1 and Y_2 are compact Hausdorff spaces, and if there exists an isomorphism T of $C(Y_1)$ onto $C(Y_2)$ with $\|T\|\,\|T^{-1}\| < 2$, then Y_1 and Y_2 are homeomorphic. In this note, we use this theorem to prove an analogous result for L^1 spaces. (Concerning the terminology "regular set isomorphism" as it is used in this paper, the reader is referred to [6].)

THEOREM. *Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be σ-finite measure spaces. If there exists an isomorphism T of $L^1(X_1, \Sigma_1, \mu_1)$ onto $L^1(X_2, \Sigma_2, \mu_2)$ satisfying $\|T\|\,\|T^{-1}\| < 2$, then there exists a regular set isomorphism Φ of (X_1, Σ_1, μ_1) onto (X_2, Σ_2, μ_2).*

PROOF. Since the measure spaces are σ-finite, the dual space of $L^1(X_i, \Sigma_i, \mu_i)$ is $L^\infty(X_i, \Sigma_i, \mu_i)$, $i = 1, 2$ [4, p. 289]. Hence the adjoint transformation T^* is an isomorphism of $L^\infty(X_2, \Sigma_2, \mu_2)$ onto $L^\infty(X_1, \Sigma_1, \mu_1)$ satisfying $\|T^*\|\,\|T^{*-1}\| < 2$. Now $L^\infty(X_i, \Sigma_i, \mu_i)$ is isometrically isomorphic to $C(Y_i)$, $i = 1, 2$, under the map $\rho_i(f) = \hat{f}$, where Y_i is the maximal ideal space of $L^\infty(X_i, \Sigma_i, \mu_i)$ and ρ_i is the Gelfand representation of $L^\infty(X_i, \Sigma_i, \mu_i)$, ([4, p. 445] or [5, p. 17]). Define a map R of $C(Y_2)$ to $C(Y_1)$ by $R(\hat{f}) = \rho_1 \circ T^* \circ \rho_2^{-1}(\hat{f})$, for $\hat{f} \in C(Y_2)$. Then clearly R is an isomorphism of $C(Y_2)$ onto $C(Y_1)$ with $\|R\|\,\|R^{-1}\| < 2$.

It thus follows that there exists a homeomorphism τ mapping Y_1 onto Y_2. And, being a homeomorphism, τ carries the clopen sets of Y_1 onto the clopen sets of Y_2. Now if $A_i \in \Sigma_i$, then $\hat{\chi}_{A_i}$ is the characteristic function of a clopen subset U_{A_i} of Y_i, and every clopen subset U of Y_i is of the form U_{A_i}, for some $A_i \in \Sigma_i$, $i = 1, 2$ [5, p. 17]. Let Φ be the map from Σ_1 to Σ_2, defined modulo null sets by $\Phi(A_1) = A_2$ if $\tau(U_{A_1}) = U_{A_2}$, where $U_{A_i} \subseteq Y_i$ and is related to $A_i \in \Sigma_i$ as in the previous sentence.

If $\mathfrak{N}_i$ denotes the family of null sets in Σ_i, then for $i = 1, 2$, $\Sigma_i/\mathfrak{N}_i$ is isomorphic as a Boolean algebra with the clopen subsets of Y_i, under the correspondence $A_i \leftrightarrow U_{A_i}$. Moreover, the Boolean supremum of a sequence U_{A_i} of clopen sets is the topological closure of the point set union of the U_{A_i}. Then, since the homeomorphism τ of Y_1 onto Y_2 preserves both point set unions and topological closures, it clearly effects an order isomorphism between the Boolean algebras of clopen sets

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of the Y_i, from which it readily follows that the map Φ of the previous paragraph is a regular set isomorphism.

REMARKS AND PROBLEMS. (a) The condition $\|T\| \|T^{-1}\| < 2$ in our theorem cannot be removed to allow for arbitrary isomorphisms of $L^1(X_1, \Sigma_1, \mu_1)$ onto $L^1(X_2, \Sigma_2, \mu_2)$ as the following example shows. Let (X_1, Σ_1, μ_1) be the measure space where $X_1 = [0, 1]$, Σ_1 is the σ-field of Lebesgue subsets of [0, 1] and μ_1 is Lebesgue measure. Let (X_2, Σ_2, μ_2) be defined as follows: $X_2 = [0, 1] \cup \{2\}$, Σ_2 consists of the Lebesgue measurable subsets of X_2, and μ_2 is the sum of Lebesgue measure on Σ_2 and of the unit point mass concentrated at 2. For each $k = 0, 1, 2, \ldots$, let I_k be the subset of [0, 1] defined by $I_k = [(2^k - 1)/2^k, (2^{k+1} - 1)/2^{k+1})$. We define a map T from $L^1(X_1, \Sigma_1, \mu_1)$ to $L^1(X_2, \Sigma_2, \mu_2)$ by

$$(T(f))(2) = \int_{I_0} f(t)\, dt$$

and

$$(T(f))(t) = f(t) - 2^{k+1}\int_{I_k} f(t)\, dt + 2^{k+1}\int_{I_{k+1}} f(t)\, dt$$

for $f \in L^1(X_1, \Sigma_1, \mu_1)$ and $t \in I_k$. ($T(f)$ has not been defined at 1, but since we are actually defining a map of equivalence classes rather than functions, the value of $(T(f))(1)$ is of no concern.) It is clear that T is linear. It is moreover one-one and surjective since given $g \in L^1(X_2, \Sigma_2, \mu_2)$, the element $f \in L^1(X_1, \Sigma_1, \mu_1)$ defined by

$$f(t) = g(t) - 2\int_{I_0} g(t)\, dt + 2 \cdot g(2)$$

for $t \in I_0$, and

$$f(t) = g(t) - 2^{k+1}\int_{I_k} g(t)\, dt + 2^{k+1}\int_{I_{k-1}} g(t)\, dt$$

for $t \in I_k$, $k > 0$, is such that $T(f) = g$. Thus T is a continuous isomorphism of $L^1(X_1, \Sigma_1, \mu_1)$ onto $L^1(X_2, \Sigma_2, \mu_2)$. However, since (X_2, Σ_2, μ_2) contains an atom while (X_1, Σ_1, μ_1) does not, there can exist no regular set isomorphism of (X_1, Σ_1, μ_1) onto (X_2, Σ_2, μ_2).

(b) It is known (see [3]) that for the theorem mentioned in the first paragraph of this article, 2 can be replaced by no larger number in the condition $\|T\| \|T^{-1}\| < 2$. Is 2 also the "best" number for a theorem of the type obtained in this paper?

(c) Can a theorem analogous to the one of this article be established for L^p, $1 < p < \infty, p \neq 2$?

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