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SYSTEMS OF PARTIAL DIFFERENTIAL OPERATORS WITH FUNDAMENTAL SOLUTIONS SUPPORTED BY A CONE

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ABSTRACT. Necessary and sufficient conditions are given for a system of partial differential operators to have a fundamental solution supported by a convex salient cone. As a simple application an overdetermined Cauchy problem is solved.

If A is a subset of $\mathbf{R}^n$ and $\mathfrak{F}$ is a space of distributions on $\mathbf{R}^n$ we denote by $\mathfrak{F}_A$ the space of distributions in $\mathfrak{F}$ which have supports contained in A. We denote by $\mathfrak{D}'$ the space of all distributions on $\mathbf{R}^n$, by $\mathfrak{S}'$ the space of temperate distributions, and by $\mathfrak{E}$ the space of infinitely differentiable functions on $\mathbf{R}^n$. If Γ is a closed convex cone in R^n with vertex at the origin, we denote by Γ^+ the dual cone defined by $\Gamma^+ = \{\xi \in R^n | \langle \xi, x \rangle \geqslant 0, x \in \Gamma\}$. Then $\Gamma^{++} = \Gamma$. The interior Γ_0^+ of Γ^+ is nonempty if and only if Γ is salient, i.e. contains no subspace other than $\{0\}$. If Γ is salient then $\mathfrak{D}'_\Gamma$ is a commutative ring relative to convolution. If H is a closed half-space with interior normal $\eta \in \Gamma_0^+$ then $\mathfrak{D}'_H$ is a $\mathfrak{D}'_\Gamma$-module, and differentiation commutes with convolution in the usual fashion. Finally we note $\mathfrak{S}'_\Gamma$ is a subring of $\mathfrak{D}'_\Gamma$. This fact is proved in the appendix below.

Let $P(z)$ be a $p \times q$ matrix over $C[z_1, \ldots, z_n]$ and denote by $P(D)$ the system of partial differential operators obtained by replacing z_j in $P(z)$ by $\partial/\partial x_j$. If $p \leqslant q$ then a fundamental solution for $P(D)$ is a $q \times p$ matrix K over $\mathfrak{D}'$ such that

$$P(D)K = \delta I \tag{1}$$

where I is the $p \times p$ identity matrix and δ is the Dirac measure at 0. In case $p = q$ then $P(D)$ has a fundamental solution with support in the closed convex salient cone Γ if and only if $P(D)$ is hyperbolic with respect to each direction in Γ_0^+, **[1]**. In case $p = q = 1$ then $P(D)$ has a *temperate* fundamental solution with support in the closed convex salient cone Γ if and only if $P(z) \neq 0$ for each z in $\Gamma_0^+ + iR^n$. This fact may be proved by means of an elementary inequality for polynomials, as is done in the introduction to **[9]**. The temperate case with $p = 1$, $q > 1$ is also considered in (**[8]**, **[9]**) and may easily be generalized as is done below. In this note we will give a sufficient, and in case Γ is semialgebraic, necessary condition for $P(D)$ in the case $p \leqslant q$ to have a fundamental solution K with support in the closed convex salient cone Γ. Our methods do not apply in the nonsalient case. The scalar case $p = q = 1$ with Γ nonsalient has been considered by A. Enqvist in **[3]** and in the temperate case in **[4]**. We will prove the following two theorems.

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THEOREM 1. *Let* Γ *be a closed convex salient cone and* $p \leq q$. *Then* $P(D)$ *admits a temperate fundamental solution* K *with support in* Γ *if and only if for each* z *in the tube* $\Gamma_0^+ + iR^n$ *the matrix* $P(z)$ *has rank* p.

THEOREM 2. *Let* Γ *be a closed convex salient cone and* $p \leq q$. *Then* $P(D)$ *admits a fundamental solution* K *with support in* Γ *if and, in the case* Γ *is semialgebraic, only if there exists a convex open set* U *in* Γ_0^+ *such that* $tU \subseteq U$ *for* $t \geq 1$, $\Gamma_0^+ = \bigcup tU$ $(t > 0)$ *and for each* z *in the tube* $U + iR^n$ *the matrix* $P(z)$ *has rank* p.

In the case $p = q$ we may dispense with the hypothesis that Γ is semialgebraic. There are at least two ways to do this. If $P(D)$ has a fundamental solution with support in Γ then the determinant $\det P(D)$ is hyperbolic with respect to each direction in Γ_0^+. From the theory of scalar hyperbolic operators [5] it follows that $\det P(D)$ is hyperbolic with respect to each direction in an open convex semialgebraic cone which contains Γ_0^+. By the lemma below we then obtain a closed convex semialgebraic cone $\Gamma' \subseteq \Gamma$ such that $P(D)$ has a fundamental solution with support in Γ'. Alternately, if Γ is not assumed semialgebraic a modification of the proof of necessity produces an open set U with the required properties other than convexity. In the case $p = q$, S. Bochner's theorem on tubes [6, Theorem 2.5.10] then shows we may replace U by its convex hull.

We first reduce the $p \times q$ system to a $1 \times N$ system, $N = \binom{q}{p}$. The notation $|J| = p$ will mean that $J = (j_1, \ldots, j_p)$ where the j_k are integers and $1 \leq j_k \leq q$ for each k. For each such J let $P^J(z)$ be the $p \times p$ matrix whose kth column is the j_kth column of $P(z)$ and let $Q_J(z)$ be the determinant of $P^J(z)$.

LEMMA. *Let* Γ *be a closed convex salient cone and* $p \leq q$. *Then* $P(D)$ *admits a fundamental solution* (*respectively, a temperate fundamental solution*) *with support in* Γ *if and only if there exist distributions* (*respectively, temperate distributions*) L_J, $|J| = p$, *with supports in* Γ *such that*

$$\sum_{|J|=p}{}' \, Q_J(D) L_J = \delta. \tag{2}$$

Here the prime over the summation symbol indicates that we sum only over p-indices $J = (j_1, \ldots, j_p)$ with $1 \leq j_1 < \cdots < j_p \leq q$. For the proof, suppose first that (2) holds with $\operatorname{supp} L_J \subseteq \Gamma$. Let $Q_J^{ik}(z)$ be the (i, k)-cofactor of $P^J(z)$, that is $(-1)^{i+k}$ times the determinant of the matrix obtained from $P^J(z)$ by removing the ith row and the kth column. Then

$$\sum_{h=1}^{p} P_{ij_h}(z) Q_J^{lh}(z) = \begin{cases} Q_J(z) & \text{if } l = i, \\ 0 & \text{if } l \neq i, \end{cases}$$

where $J = (j_1, \ldots, j_p)$. If we set

$$K_{jl} = \sum_{h=1}^{p} \sum{}' \, Q_J^{lh}(D) L_J, \qquad 1 \leq j \leq q,\ 1 \leq l \leq p,$$

where the inner sum is over $|J| = p$ such that $j_h = j$, then $\operatorname{supp} K_{jl} \subseteq \Gamma$ and

$$\sum_{j=1}^{q} P_{ij}(D) K_{jl} = \sum_{|J|=p}{}' \sum_{h=1}^{p} P_{ij_h}(D) Q_J^{lh}(D) L_J$$

whence (1) follows. If the L_J are temperate, then so also are the K_{jl}.

Conversely suppose (1) holds with supp $K_{jl} \subseteq \Gamma$. Let $A_{ij} = P_{ij}(D)\delta$ so $A * K = \delta I$. Since the distributions with supports in Γ form a commutative ring with respect to convolution it makes sense to take the determinant. From the Binet-Cauchy formula we obtain

$$\delta = \det(A * K) = \sum_{|J|=p}{}' (\det A_J) * (\det K_J)$$

where A_J is the matrix whose kth column is the j_kth column of A and K_J is the matrix whose kth row is the j_kth row of K. Since $A_J = P^J(D)(\delta I)$ we see that $\det A_J = Q_J(D)\delta$. If we set $L_J = \det K_J$ then (2) follows and $\operatorname{supp} L_J \subseteq \Gamma$. If K is temperate then the L_J are temperate (see Appendix). Note it is not difficult to see if we start with K and set $L_J = \det K_J$ then the construction at the beginning of the proof yields the original K.

The lemma is now proved and moreover Theorem 1 follows from the $p = 1$ case which is considered in [**8**], [**9**]. The proof of the lemma is quite standard. The argument for example is similar to the argument in the $p = q$ case given in [**1**, Lemma 3.2]. The sufficiency of (2) in the $p < q$ case is the same as the argument in [**11**, Theorem 4.1]. We gave the argument, however, because prior to proceeding to the proof of Theorem 2 we will use the notation and proof of the lemma to solve an overdetermined Cauchy problem for a half-space when compatibility conditions are satisfied. Let $P^t(z)$ denote the transpose of the matrix $P(z)$.

THEOREM 3. *Let Γ be a closed convex salient cone and $p < q$. Assume* (1) *holds with* supp $K \subseteq \Gamma$. *Let $\eta \in \Gamma_0^+$ and let H be the closed half-space $\{x \in R^n | \langle x, \eta\rangle > 0\}$. If $w \in (\mathfrak{D}')^p$ and if* $\operatorname{supp}(P^t(D)w) \subseteq H$ *then there exists a unique $u \in (\mathfrak{D}')^p$ such that*

$$\operatorname{supp} u \subseteq H, \quad P^t(D)u = P^t(D)w.$$

Moreover, if $w \in \mathcal{E}^p$ then $u \in \mathcal{E}^p$.

We prove uniqueness first. Suppose $u \in (\mathfrak{D}'_H)^p$ and let $v = K^t * P^t(D)u$. Since $K_{jk} \in \mathfrak{D}'_\Gamma$ we have

$$v_k = \sum_j K_{jk} * \sum_h P_{hj}(D)u_h$$

$$= \sum_{j,h} P_{hj}(D)K_{jk} * u_h = u_k.$$

Thus $u = K^t * P^t(D)u$ for any $u \in (\mathfrak{D}'_H)^p$ which gives the uniqueness.

For existence we define $u \in (\mathfrak{D}'_H)^p$ by $u = K^t * P^t(D)w$. Note if w is smooth, then so is u which gives the last part. To see that u is a solution, since we have no control over supp w some care is required in commuting convolutions and differentiations. By the proof of the lemma we have $L_J \in \mathfrak{D}'_\Gamma$ such that

$$K_{jk} = \sum_{h=1}^{p} \sum{}' Q_J^{kh}(D)L_J$$

where the inner sum is over $|J| = p$ with $j_h = j$. Then, since supp $L_J \subseteq \Gamma$,

$$\begin{aligned} u_k &= \sum_j K_{jk} * P_{ij}(D)w_i \\ &= \sum_{|J|=p}' \sum_h Q_J^{kh}(D)L_J * \sum_i P_{ij_h}(D)w_i \\ &= \sum_{|J|=p}' L_J * \sum_{i,h} Q_J^{kh}(D)P_{ij_h}(D)w_i. \end{aligned}$$

Now

$$Q_J(D)w_k = \sum_{i,h} Q_J^{kh}(D)P_{ij_h}(D)w_i$$

implies $Q_J(D)w_k$ has support in H for each J and each k. From the above computation we have

$$u_k = \sum_{|J|=p}' L_J * Q_J(D)w_k$$

and therefore

$$\begin{aligned} \sum_k P_{kl}(D)u_k &= \sum_{|J|=p}' L_J * Q_J(D)\sum_k P_{kl}(D)w_k \\ &= \sum_{|J|=p}' Q_J(D)L_J * \sum_k P_{kl}(D)w_k \\ &= \sum_k P_{kl}(D)w_k \end{aligned}$$

where the first equality follows from the fact that $Q_J(D)w_k$ has support in H and the second from the fact that $P^t(D)w$ has support in H.

Proof of Theorem 2. By the lemma we may assume $p = 1$. Thus $P(z) = (P_1(z), \ldots, P_q(z))$. Suppose first that $P_1(z), \ldots, P_q(z)$ have no common zero in $U + iR^n$ where U is a convex open subset of Γ_0^+ such that $tU \subseteq U$ if $t > 1$ and Γ_0^+ is the union of tU for $t > 0$. Locally in $U + iR^n$ we can find holomorphic functions F_j such that $\sum P_j(z)F_j(z) = 1$. By Cartan's Theorem B [**6**, Theorem 7.4.3] these local solutions may be modified to fit together to give global holomorphic functions F_j (here we use the convexity of U). Moreover by [**8**, Theorem 1] we may choose the holomorphic functions F_j so that

$$|F_j(z)| \leq C(1 + |z|)^N d(\xi)^{-m}, \qquad z \in U + iR^n,$$

for some constants C, N and m. Here ξ is the real part of z and $d(\xi)$ is the minimum of 1 and the distance from ξ to the boundary of U. By [**10**, Proposition 6, p. 306] F_j is the Laplace transform of a distribution K_j. Then $\sum P_j(D)K_j = \delta$ and it remains to locate the support of K_j. That supp K_j is contained in Γ follows directly by estimating

$$\langle K_j, \phi \rangle = (2\pi)^{-n} \int F_j(\xi + i\eta)\hat{\phi}(i\xi - \eta)\,d\eta$$

where $\phi \in \mathfrak{S}$ has support in a compact convex set disjoint from Γ and $\hat{\phi}$ is the Fourier transform of ϕ. The integral is independent of $\xi \in \Gamma_0^+$ and we simply separate Γ and supp $\hat{\phi}$ by a hyperplane with normal $\xi \in \Gamma_0^+$ and let $|\xi| \to \infty$. Alternately supp K_j is contained in Γ by [**10**, Remark 1, p. 310].

For the converse we modify the argument in [**1**, Theorem 3.5]. Assume there exist $K_j \in \mathfrak{D}'_\Gamma$ such that $\sum P_j(D)K_j = \delta$. Choose $\phi \in \mathfrak{S}$ with $\phi(x) = 1$ if $|x| < 1$ and $\phi(x) = 0$ if $|x| > 2$. Then $\sum P_j(D)(\phi K_j) = \delta + g$ where $g \in \mathfrak{D}'$ and supp $g \subseteq \{x \in \Gamma | 1 \leq |x| \leq 2\}$. Let G_j be the Laplace transform of ϕK_j and let G be the Laplace transform of g. Then G and the G_j are entire functions and $\sum P_j(z)G_j(z) = 1 + G(z)$. By the Paley-Wiener theorem [**2**, p. 211]

$$|G(z)| \leq C(1 + |z|)^N e^{h(-\xi)} \tag{3}$$

where $z = \xi + i\eta$ and where $h(-\xi) = \sup\{-\langle \xi, x\rangle | x \in \Gamma, 1 \leq |x| \leq 2\}$. If $\xi \in \Gamma_0^+$ then $\langle \xi, x\rangle > 0$ for each $x \in \Gamma$, $x \neq 0$ and so $h(-\xi) = -\text{dist}(\xi, \partial\Gamma_0^+)$. Here $\text{dist}(\xi, \partial\Gamma_0^+) = \inf\{\langle \xi, x\rangle | x \in \Gamma, |x| = 1\}$ is easily seen to be the distance from ξ to the boundary of Γ_0^+. At any common zero of the P_j we have $G(z) = -1$. Thus for some constants C and N we have

$$\text{dist}(\xi, \partial\Gamma_0^+) \leq C + N \log(1 + |z|) \tag{4}$$

if $\xi \in \Gamma_0^+$, $z = \xi + i\eta$ and $P_j(z) = 0, j = 1, \ldots, q$.

Suppose now Γ is semialgebraic. First note Γ_0^+ is the complement of the projection on the first n coordinates of the set of (ξ, x) such that $\xi \in R^n$, $x \in \Gamma$, $x \neq 0$, $\langle \xi, x\rangle < 0$ and hence is semialgebraic by the Seidenberg-Tarski theorem. It follows that the set of (μ, ξ, x) such that $\xi \in \Gamma_0^+$, $x \in \Gamma$, $|x| = 1$, $\mu > \langle \xi, x\rangle$ is semialgebraic and hence by the Seidenberg-Tarski theorem the set M of (μ, ξ) such that $\xi \in \Gamma_0^+$ and $\mu > \text{dist}(\xi, \partial\Gamma^+)$ is semialgebraic. An application of the Seidenberg-Tarski theorem shows that the closure and interior of a semialgebraic set is semialgebraic. Thus $\partial M \cap (R \times \Gamma_0^+) = \{(\mu, \xi) | \xi \in \Gamma_0^+, \mu = \text{dist}(\xi, \partial\Gamma^+)\}$ is semialgebraic. This property of the distance function, that the graph is semialgebraic, is known in other cases as well but is particularly simple to prove in our case because we have a nice formula for the distance to the boundary of a convex cone. It now follows that the set L_0 of (μ, τ, ξ, η) such that $\xi \in \Gamma_0^+$, $\mu = \text{dist}(\xi, \partial\Gamma^+)$, $\tau \geq |\xi + i\eta|$, $P_j(\xi + i\eta) = 0, j = 1, \ldots, q$, is semialgebraic. Again by the Seidenberg-Tarski theorem the projection L on the first two coordinates is semialgebraic. By (4) if $(\mu, \tau) \in L$ then $\mu \leq C + N\log(1 + \tau)$. By [**5**, Lemma 2.1, p. 276] it follows that there is a constant C_1 such that $\mu \leq C_1$ if $(\mu, \tau) \in L$. Now let $U = \{\xi \in \Gamma_0^+ | \text{dist}(\xi, \partial\Gamma^+) > C_1\}$.

APPENDIX. We now show $\mathfrak{S}'_\Gamma$ is a subring of $\mathfrak{D}'_\Gamma$. Suppose $f, g \in \mathfrak{S}'_\Gamma$. Then

$$e^{-\langle \xi, \cdot\rangle}(f * g) = (e^{-\langle \xi, \cdot\rangle} f) * (e^{-\langle \xi, \cdot\rangle} g).$$

For each $\xi \in \Gamma_0^+$ the factors on the right are in $\mathfrak{S}'$ and therefore by [**10**, Corollary, p. 302] are in $\mathcal{O}'_C$. It follows that $e^{-\langle \xi, \cdot\rangle}(f * g)$ is in $\mathcal{O}'_C$. Taking Fourier transforms we obtain

$$(e^{-\langle \xi, \cdot\rangle}(f * g))^\wedge(\eta) = F(\xi + i\eta)G(\xi + i\eta)$$

where F (respectively G) is the Fourier transform of f (respectively g). The product on the right is a holomorphic function in $\Gamma_0^+ + iR^n$ and by [**7**, Theorem 1] is the Laplace transform of a distribution $u \in \mathfrak{S}'_\Gamma$. Obviously $u = f * g$.

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