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INTERPOLATING FUNCTIONS ASSOCIATED WITH SECOND-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Functions are exhibited which interpolate the magnitude of a solution y of a linear, homogeneous, second-order differential equation at its critical points, $|y'|$ at the zeros of y, and $|\int_{x_0}^{x} y(t)h(t)\,dt|$ at the zeros of y. Except for a normalization condition, the interpolating functions are independent of the specific solution y. A theorem similar in its conclusions to the Sonin-Pólya-Butlewski theorem is presented and examples are given.

1. Introduction. Suppose y_1 and y_2 form a fundamental system for the differential equation

$$(p(x)y')' + q(x)y = 0, \qquad a < x < b, \tag{1}$$

where we assume throughout that p and q are continuous and $p > 0$ on (a, b). Let $y = Ay_1 + By_2$ be an arbitrary nontrivial solution of (1), normalized so that $A^2 + B^2 = 1$. We will exhibit a function, independent of A and B, which interpolates $|y|$ at the critical points of y, and others, also independent of A and B, which interpolate $|y'|$ and $|\int_{x_0}^{x} y(t)h(t)\,dt|$ at the zeros of y.

2. Preliminary lemmas.

LEMMA 1. *Suppose f_1, f_2, g_1 and g_2 are real numbers, with $f_1^2 + f_2^2 > 0$. Let*

$$f = Af_1 + Bf_2, \qquad g = Ag_1 + Bg_2, \tag{2}$$

with

$$A^2 + B^2 = 1. \tag{3}$$

Then

$$|g| = |f_1 g_2 - f_2 g_1| / \left(f_1^2 + f_2^2\right)^{1/2} \quad \text{if } f = 0. \tag{4}$$

PROOF. The conclusion follows from the identity

$$(A^2 + B^2)(f_1 g_2 - f_2 g_1)^2 = (fg_2 - f_2 g)^2 + (f_1 g - fg_1)^2, \tag{5}$$

which can be derived by solving (2) for A and B, squaring and adding, and then multiplying both sides of the resulting equation by $(f_1 g_2 - f_2 g_1)^2$. Although this seemingly requires the assumption that $f_1 g_2 - f_2 g_1 \neq 0$, it is easily verified that (4) holds for all values of the quantities appearing in it. Invoking (3) yields (5).

In the next two lemmas y_1 and y_2 need not be solutions of (1).

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LEMMA 2. *Let* y_1 *and* y_2 *be differentiable and suppose their Wronskian,* $W = y_1y_2' - y_2y_1'$, *has no zeros in* (a, b). *Let*

$$y = Ay_1 + By_2 \qquad (A^2 + B^2 = 1). \tag{6}$$

Then

$$|y(\bar{x})| = |W(\bar{x})|/\left[(y_1'(\bar{x}))^2 + (y_2'(\bar{x}))^2\right]^{1/2} \quad \text{if } \bar{x} \in (a, b) \text{ and } y'(\bar{x}) = 0, \tag{7}$$

and

$$|y'(\bar{x})| = |W(\bar{x})|/\left[y_1^2(\bar{x}) + y_2^2(\bar{x})\right]^{1/2} \quad \text{if } \bar{x} \in (a, b) \text{ and } y(\bar{x}) = 0. \tag{8}$$

PROOF. To obtain (7), take $f_i = y_i'(\bar{x})$ and $g_i = y_i(\bar{x})$ $(i = 1, 2)$ in Lemma 1. To obtain (8), take $f_i = y_i(\bar{x})$ and $g_i = y_i'(\bar{x})$ in Lemma 1. The nonvanishing of W guarantees that the denominators in (7) and (8) are nonzero.

LEMMA 3. *Suppose* y_1h *and* y_2h *are locally integrable and* $y_1^2 + y_2^2 > 0$ *on* (a, b). *Define*

$$s(x) = \int_{x_0}^{x} [y_1(t)y_2(x) - y_2(t)y_1(x)]h(t)\,dt, \qquad a < x_0, x < b, \tag{9}$$

and let y *satisfy* (6). *Then*

$$\left|\int_{x_0}^{\bar{x}} y(t)h(t)\,dt\right| = \frac{|s(\bar{x})|}{\left[y_1^2(\bar{x}) + y_2^2(\bar{x})\right]^{1/2}} \quad \text{if } \bar{x} \in (a, b) \text{ and } y(\bar{x}) = 0. \tag{10}$$

The conclusion also holds with $x_0 = a$ *if* $\int_a y_i(t)h(t)\,dt$ *exists* $(i = 1, 2)$, *or with* $x_0 = b$ *if* $\int^b y_i(t)h(t)\,dt$ *exists* $(i = 1, 2)$.

PROOF. Take $f_i = y_i(\bar{x})$ and $g_i = \int_{x_0}^{\bar{x}} y_i(t)h(t)\,dt$ $(i = 1, 2)$ in Lemma 1.

3. Main results. If y_1 and y_2 are linearly independent solutions of (1), then

$$y_1y_2' - y_2y_1' = k/p \qquad (k = \text{constant} \neq 0). \tag{11}$$

This and Lemma 2 yield the following theorem.

THEOREM 1. *Let* y_1 *and* y_2 *be solutions of* (1) *satisfying* (11), *and suppose* y *satisfies* (6). *Then the function*

$$I_1 = |k|/p\left[(y_1')^2 + (y_2')^2\right]^{1/2} \tag{12}$$

interpolates $|y|$ *at the critical points of* y, *and the function*

$$I_2 = |k|/p\left(y_1^2 + y_2^2\right)^{1/2}$$

interpolates $|y'|$ *at the zeros of* y.

This theorem can essentially be obtained by applying known transformations to (1), although the argument presented above is simpler and appears to require fewer assumptions. The substitutions of

$$t = \int^x \frac{du}{p(u)}, \qquad Y(t) = y(x), \tag{13}$$

transform (1) into

$$d^2Y/dt^2 + Q(t)Y = 0, \tag{14}$$

where $Q(t) = p(x)q(x)$. If y_1 and y_2 are as defined at the beginning of this section and $Y_i(t) = y_i(x)$ $(i = 1, 2)$, let $f(t) = Y_1^2(t) + Y_2^2(t)$. L. Lorch and P. Szegö [3] have shown that the substitutions

$$z = \int^t \frac{dv}{f(v)}, \qquad u(z) = (f(t))^{1/2}Y(t)$$

transform (14) into $d^2u/dz^2 + k^2u = 0$. Therefore, the general solution of (1) is

$$y(x) = C\left[y_1^2(x) + y_2^2(x)\right]^{1/2} \sin\left(k\int^t \frac{du}{f(u)}\right), \tag{15}$$

with t related to x as in (13). (This formula was given by Borůvka [**1**, p. 43] for the case where $p(x) = 1$.) By differentiating (15) while recalling (13), and noting that $y(x) = 0$ if and only if $\sin(k\int^t du/f(u)) = 0$ (and so $\cos(k\int^t du/f(u)) = \pm 1$), it can be shown that a constant multiple of I_2 interpolates $|y'|$ at the zeros of y.

Formally, the equation $(u'/q)' + u/p = 0$ has solutions $u = py'$, where y satisfies (1). Applying the procedure of the last paragraph to this equation leads to a general formula for y' (also given by Borůvka for the case where $p(x) = 1$ [**1**, p. 43]), from which it can be shown that a constant multiple of I_1 interpolates $|y|$ at the zeros of y'. It would appear that this derivation of I_1 requires the additional assumption that p has no zeros on (a, b).

It should be observed that the use of the function $y_1^2 + y_2^2$ for obtaining qualitative properties of solutions of (1) occurs in many places in the literature; for examples, see [**2**, pp. 515–519], [**4**], [**5**] and [**6**].

THEOREM 2. *Suppose* (1) *has linearly independent solutions* y_1 *and* y_2 *such that the function* $F = q(y_1^2 + y_2^2)'$ *does not change sign on* (a, b). *Let* $\{x_n\}$ *be an increasing sequence of critical points of an arbitrary nontrivial solution* y *of* (1). *Then the sequence* $\{|y(x_n)|\}$ *is nondecreasing if* $F \geqslant 0$, *or nonincreasing if* $F \leqslant 0$. *The monotonicity of* $\{|y(x_n)|\}$ *is strict if* F *is not identically zero on any subinterval of* (a, b).

PROOF. Assume without loss of generality that y is normalized as in (6); then Theorem 1 implies that $|y(x_n)| = I_1(x_n)$, with I_1 as in (12). Differentiating I_1 and noting that y_1 and y_2 satisfy (1) yields $I_1' = pFI_1^3/2k^2$. This implies the conclusion.

THEOREM 3. *Suppose* y_1 *and* y_2 *are solutions of* (1) *which satisfy* (11), *that* $a < x_0 < b$, *and that* s *satisfies*

$$(p(x)s')' + q(x)s = kh(x), \qquad s(x_0) = s'(x_0) = 0,$$

where h *is continuous on* (a, b). *Then* (10) *holds, with* y *as in* (6). *This is also true with* $x_0 = a$ *if* $\int_a y_i(t)h(t)\,dt$ *exists* $(i = 1, 2)$, *or with* $x_0 = b$ *if* $\int^b y_i(t)h(t)\,dt$ *exists* $(i = 1, 2)$.

PROOF. By variation of parameters, s is as in (9), so Lemma 3 implies the conclusion.

4. Examples. Let (1) be Bessel's equation,

$$(xy')' + \frac{1}{x}(x^2 - \nu^2)y = 0,$$

with $y_1 = J_\nu$ and $y_2 = Y_\nu$, the Bessel functions of the first and second kinds, and let

$$\mathcal{C}_\nu = AJ_\nu + BY_\nu \qquad (A^2 + B^2 = 1) \tag{16}$$

be an arbitrary normalized cylinder function.

EXAMPLE 1. Since the Wronskian of y_1 and y_2 in this case is

$$W(x) = J_\nu(x)Y_\nu'(x) - Y_\nu(x)J_\nu'(x) = 2/\pi x \tag{17}$$

[8, p. 76], Theorem 1 implies that

$$|\mathcal{C}_\nu(\bar{x})| = \frac{2}{\pi\bar{x}\left[(J_\nu'(\bar{x}))^2 + (Y_\nu'(\bar{x}))^2\right]^{1/2}} \quad \text{if } \bar{x} > 0 \text{ and } \mathcal{C}_\nu'(\bar{x}) = 0,$$

and

$$|\mathcal{C}_\nu'(\bar{x})| = \frac{2}{\pi\bar{x}\left[(J_\nu(\bar{x}))^2 + (Y_\nu(\bar{x}))^2\right]^{1/2}} \quad \text{if } \bar{x} > 0 \text{ and } \mathcal{C}_\nu(\bar{x}) = 0. \tag{18}$$

It is perhaps worth noting that both interpolating functions are rational if $\nu = k + 1/2$, where k is an integer [8, p. 297]. Since $x(J_\nu^2 + Y_\nu^2)$ increases if $\nu > 1/2$ and decreases if $\nu < 1/2$ [8, p. 466], (18) implies that if $\{x_n\}$ is an increasing sequence of positive zeros of $\mathcal{C}_\nu$, then the sequence $\{|\mathcal{C}_\nu'(x_n)|\}$ decreases if $\nu > 1/2$ and increases if $\nu < 1/2$. Since $J_\nu^2 + Y_\nu^2$ decreases for all ν [8, p. 466], (18) also implies that the sequence $\{x_n|\mathcal{C}_\nu'(x_n)|\}$ increases for all ν.

EXAMPLE 2. If $\mathcal{C}_\nu$ is a cylinder function as in (16), then $\int_0^x t^\mu \mathcal{C}_\nu(t)\,dt$ exists if $\mu \pm \nu > -1$. From (11) and (17), the initial value problem associated with this integral according to Theorem 3 is

$$(xs')' + \frac{1}{x}(x^2 - \nu^2)s = 2x^\mu/\pi, \qquad s(0) = s'(0) = 0,$$

which has the solution $s = 2s_{\mu\nu}/\pi$, where $s_{\mu\nu}$ is Lommel's function of the first kind [8, pp. 345–346]. Now (10) implies that

$$\left|\int_0^{\bar{x}} t^\mu \mathcal{C}_\nu(t)\,dt\right| = \frac{2|s_{\mu\nu}(\bar{x})|/\pi}{\left[J_\nu^2(\bar{x}) + Y_\nu^2(\bar{x})\right]^{1/2}} \quad \text{if } \bar{x} > 0 \text{ and } \mathcal{C}_\nu(\bar{x}) = 0.$$

EXAMPLE 3. If $\mu < 1/2$ and $\mathcal{C}_\nu$ is as in (16), then $\int_x^\infty t^\mu \mathcal{C}_\nu(t)\,dt$ exists for $x > 0$. The initial value problem associated with this integral according to Theorem 3 is

$$(xs')' + \frac{1}{x}(x^2 - \nu^2)s = 2x^\mu/\pi, \qquad s(\infty) = s'(\infty) = 0,$$

which has the solution $s = 2S_{\mu\nu}/\pi$, where $S_{\mu\nu}$ is Lommel's function of the second kind [8, p. 347]. Therefore, Theorem 3 implies that

$$\left|\int_{\bar{x}}^{\infty} t^{\mu}\mathcal{C}_{\nu}(t)\,dt\right| = \frac{2|S_{\mu\nu}(\bar{x})|/\pi}{\left[J_{\nu}^{2}(\bar{x}) + Y_{\nu}^{2}(\bar{x})\right]^{1/2}} \quad \text{if } \bar{x} > 0 \text{ and } \mathcal{C}_{\nu}(\bar{x}) = 0.$$

EXAMPLE 4. Theorem 2 and the fact that $J_{\nu}^{2} + Y_{\nu}^{2}$ decreases imply that the successive maxima of $|\mathcal{C}_{\nu}|$ on (ν, ∞) form a decreasing sequence. This is a known result [8, p. 488].

The conclusion of Theorem 2 is similar to that of the Sonin-Pólya-Butlewski theorem [7, p. 166], which says that if $p > 0$, $q > 0$, p and q are continuously differentiable on (a, b), and pq is monotonic, then $\{|y(x_n)|\}$ is monotonic in the opposite sense. By considering transformed versions of Bessel's equation and using known monotonicity properties of $x(J_{\nu}^{2} + Y_{\nu}^{2})$ and $(x^2 - \nu^2)^{1/2}(J_{\nu}^{2} + Y_{\nu}^{2})$ [8, p. 446], it is possible to obtain from Theorem 2 monotonicity properties of sequences of maxima of $x^{1/2}|\mathcal{C}_{\nu}|$ and $(x^2 - \nu^2)^{1/4}|\mathcal{C}_{\nu}|$; however, the differential equations in question also satisfy the hypotheses of the Sonin-Pólya-Butlewski theorem, and the results are not new. The following contrived example presents a result implied by Theorem 2 that cannot be obtained from the Sonin-Pólya-Butlewski theorem.

EXAMPLE 5. If g is positive and continuously differentiable on $(0, \infty)$, then

$$y_1(x) = x\cos\left(\int_0^x g(t)\,dt\right) \quad \text{and} \quad y_2(x) = x\sin\left(\int_0^x g(t)\,dt\right)$$

form a fundamental system for

$$\left(\frac{y'}{x^2 g}\right)' + \left(\frac{2}{x^3 g} + \frac{g'}{x^2 g^2} + \frac{g}{x}\right)y = 0, \qquad x > 0. \tag{19}$$

Clearly (19) is oscillatory if $\int^{\infty} g(t)\,dt = \infty$, and since $y_1^2(x) + y_2^2(x) = x^2$ is increasing, it satisfies the hypotheses of Theorem 2 on $(0, \infty)$ if $g' \geqslant 0$; therefore, if these assumptions hold and ϕ is any constant, then the absolute values of

$$y(x) = x\cos\left(\phi + \int_0^x g(t)\,dt\right)$$

at any increasing sequence of critical points of y form an increasing sequence. The Sonin-Pólya-Butlewski theorem does not imply this, since

$$pq = \frac{1}{x^2 g}\left(\frac{2}{x^3 g} + \frac{g'}{x^2 g^2} + \frac{g}{x}\right)$$

need not be monotonic; for example, let

$$g(x) = \int_0^x (2 - \cos e^t)\,dt.$$

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