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ON BEURLING'S THEOREM FOR LOCALLY COMPACT GROUPS

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ABSTRACT. Beurling's Theorem in spectral analysis of bounded functions on the real line is generalized to a class of semidirect products of locally compact abelian groups.

As an immediate consequence of Wiener's Tauberian Theorem one has, by duality (see [**4**, p. 181]), the following theorem on spectral analysis: Unless $f \in L_\infty(\mathbf{R})$ is zero almost everywhere, the w^*-closed subspace generated by the translates of f contains a function $e^{i\lambda x}$ for some $\lambda \in \mathbf{R}$. Beurling proved [**1**] much more about a smaller class of functions. His theorem is, essentially, the following:

BEURLING'S THEOREM. *Let f be a nonzero, bounded uniformly continuous function on $\mathbf{R}$. Then there exists a real number λ, such that the function $e^{i\lambda x}$ belongs to the w^*-closure of some norm-bounded set of linear combinations of translates of f.*

This result was generalized to locally compact abelian groups [**2**], [**3**].

The purpose of this note is to generalize Beurling's Theorem to a class of semidirect products of locally compact abelian groups.

For the group G^* of the linear transformations on the real line of the form $ax + b$, $a > 0$, $b \in \mathbf{R}$, the following result was announced in [**5**]: Every proper closed two-sided ideal of $L_1(G^*)$ is contained in a maximal modular two-sided ideal. Our result implies, by duality, that every proper closed two-sided ideal of $L_1(G^*)$ is contained, actually, in the kernel of a one-dimensional representation of G^* and that all closed two-sided maximal ideals of $L_1(G^*)$ are of this type.

Let $G = N \rtimes H$ denote the semidirect product of the groups N and H and let $h \to \tau_h$ be the homomorphism which carries H onto a group of automorphisms of N.

For a locally compact abelian group G, let $\hat{G}$ denote the character group of G and 1_G the identity element of G.

We prove the following theorem:

THEOREM. *Let $G = N \rtimes H$ where N and H are locally compact abelian groups. Suppose that for every $\chi \in \hat{N}$, there exists a sequence $\{h_k\}_{k=1}^{\infty}$ in H such that $\chi \circ \tau_{h_k} \to 1_{\hat{N}}$ in the w^*-topology of $L_\infty(N)$. Then, for every $f \in L_\infty(G)$, $f \neq 0$, the w^*-closed subspace generated by the two-sided translates of f contains a character ψ of G. Moreover, if f is uniformly continuous, the character ψ belongs to the w^*-closure of some norm-bounded set of linear combinations of two-sided translates of f.*

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PROOF. Let M denote the w^*-closed subspace generated by the two-sided translates of the function $f \in L_\infty(G), f \neq 0$. The subspace M contains all functions g such that

$$g(n, h) = f(n_2 \tau_{h''h}(n_1) \tau_{h''}(n), h'h''h) \tag{1}$$

where $n_1, n_2 \in N$ and $h', h'' \in H$.

Let $h'' = 1_H$ and $n_1 = 1_N$. Then, applying Wiener's Theorem for $N \times H$, we deduce that $\chi \otimes \Phi \in M$ for some $\chi \in \hat{N}$ and $\Phi \in \hat{H}$. (Here, $\chi \otimes \Phi$ denotes the function defined by $\chi \otimes \Phi(n, h) = \chi(n)\Phi(h)$.) If we apply (1) to $\chi \otimes \Phi \in M$ where $n_1 = n_2 = 1_N$, $h' = 1_H$ and $h'' = h_k$, we have

$$(\chi \circ \tau_{h_k}) \otimes \Phi \in M \qquad (k = 1, 2, \ldots)$$

and

$$(\chi \circ \tau_{h_k}) \otimes \Phi \xrightarrow{w^*} 1_{\hat{N}} \otimes \Phi$$

which is a character of G.

If f is uniformly continuous, then by Beurling's Theorem for $N \times H$, the function $\chi \otimes \Phi$ belongs to the w^*-closure of some norm-bounded set of linear combinations of two-sided translates of f. Proceeding as above, we complete the proof of the theorem.

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