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APPROXIMATION BY NONFUNDAMENTAL SEQUENCES OF TRANSLATES

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ABSTRACT. For functions $f(t)$ satisfying certain growth conditions, we consider a sequence of the form $\{f(c_n - t)\}$, nonfundamental in $L_2(R)$, and find a representation for those functions which are in the closure of its linear span. Some theorems concerning degree of approximation are also proved.

In [1], we found necessary and sufficient conditions for a sequence of the form $\{f(c_n - t)\}$ to be fundamental in $L_2(R)$. In this paper, motivated by earlier research of L. Schwartz [2], and I. I. Hirschmann, Jr. [3] (see also J. Korevaar [4], W. A. J. Luxemburg and J. Korevaar [5, p. 35, Theorem 8.2], and Clarkson and Erdös [6]), we consider the nonfundamental case and find a representation of those functions which are in the $L_2(R)$ closure of the linear span of $\{f(c_n - t)\}$. Our result applies to a different class of functions than those considered by the above mentioned authors. The techniques developed to attack this problem are also applied to find a lower bound for the $L_2(R)$ distance from $f(c - t)$ to the linear span of $\{f(c_r - t); r = 0, \ldots, n\}$, obtaining a result similar to [5, p. 31, Theorem 7.1], [4, p. 363, Theorem 4], or [6, p. 6, Theorem 2]. Finally, we also prove a Jackson type theorem valid for a class of continuous functions defined on a bounded interval.

In what follows, $\{d_n\}$ will be a sequence of distinct real numbers, satisfying the following conditions:

$$|c_n^2 - c_r^2| \geq \rho|n - r| \quad (\rho > 0) \quad \text{and} \quad \sum{}' |c_n|^{-2} < \infty. \tag{1}$$

(By $\Sigma'|c_n|^{-2}$ we denote the sum of all terms of the form indicated, with nonvanishing denominator.) Note that (1) is satisfied if, for instance

$$|c_{n+1}| \geq \rho|c_n| \quad (\rho \geq \sqrt{2}).$$

Given a function $f(t)$, by $F(t)$ we shall denote its Fourier transform. Thus

$$F(t) = (2\pi)^{-1/2} \int_R \exp(xti) f(t)\, dt.$$

We shall assume that there are strictly positive numbers α, a and b, such that for t real, $f(t) = O[\exp(-\alpha t^2)]$, $F(t) = O[\exp(-at^2)]$, $t \to \infty$, and $\exp(-bt^2)/F(t)$ is in $L_2(R)$. By a theorem of Babenko, later generalized by Gel'fand and Šilov, we know that the growth condition on $f(t)$ can be replaced by the assumption that F is an

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entire function of order 2 and finite type (cf. [7, p. 238, Theorem 3]). Finally, if $f_n(t) = f(c_n - t)$, and $F_n(t)$ is its Fourier transform, it is readily seen that $F_n(-t) = F(t)\exp(c_n ti)$; we shall denote by S the linear span of the sequence $\{f_n\}$, and by T the linear span of the sequence $\{F_n\}$.

Our first result is:

THEOREM 1. *Assume $\{c_n\}$ satisfies* (1). *Then if the function $g(t)$ is in the $L_2(R)$ closure of S, it coincides a.e. on R with a series of the form $\sum b_n f_n(t)$.*

Theorem 1 is proved with the help of the following auxiliary proposition:

LEMMA. *Assume $\{c_n\}$ satisfies* (1). *Then there are continuous functions $p_k(t) = p_k(t, \mu)$, having Fourier transforms $m_k(t) = m_k(t, \mu)$, satisfying the following conditions:*

(a) *Let $h(t) = \exp(-bt^2)/|F(t)|$; then for every $\mu < 1/(2b)$ and positive,*

$$|m_k(t, \mu)| \leqslant d\exp\left[-(1/(2\mu) - b)t^2 + \mu c_k^2\right]h(t),$$

where d is independent of k.

(b) $\int_R p_k(t)f_n(t)\,dx = \int_R m_k(t)F_n(t)\,dt = \delta_{kn}$ *where δ_{kn} is Kronecker's delta.*

(c) *For $g(t)$ in $L_2(R)$, let $b_k(g) = \int_R p_k(t)g(t)\,dt$, then for any $\delta < \alpha$ and positive, there is a value of μ and a number γ such that for all real t,*

$$|b_n(g)f_n(t)| \leqslant c^2\|g\|_{L_2(R)}\exp\left(-\delta c_k^2 + \gamma t^2\right),$$

where c is independent of k, and if for this value of μ, $S(g, t) = \sum b_n(g)f_n(t)$, then

$$|S(g, t)| \leqslant M(t)\|g\|_{L_2(R)}, \quad \text{where } M(t) = c\exp(\gamma t^2)\sum\exp(-\delta c_n^2).$$

Using the preceding Lemma, we can prove:

THEOREM 2. *Assume $\{c_n\}$ satisfies* (1), *and let c be any real number not in the range of the sequence $\{c_n\}$. If $|c| = |c_n|$ for some n, let $m_c = 1$; otherwise, let $m_c = \inf|1 - (c/c_n)^2|$, the infimum being taken over the set of natural numbers. Let d_c denote the $L_2(R)$ distance from $F(t)\exp(cti)$ to T. Then there is a number $D > 0$, independent of c and k, such that $d_c \geqslant Dm_c^2\exp(-c^2/8b)$.*

REMARK. Since the Fourier transform is norm-preserving in $L_2(R)$, d_c also denotes the $L_2(R)$ distance from $f(c - t)$ to S. It should also be pointed out that the lower bound in Theorem 2 is not the best possible.

From Theorem 2 we obtain the following

COROLLARY. *If c is not in the range of the sequence $\{c_n\}$, then neither $F(t)\exp(cti)$ is in the $L_2(R)$ closure of T, nor is $f(c - t)$ in the $L_2(R)$ closure of S.*

Finally, we have:

THEOREM 3. *Assume that $\{c_n\}$ satisfies* (1), *and let $g(t)$ be a function in the $L_2(R)$ closure of S. Let (a_1, b_1) be a bounded interval, assume $g(t)$ is continuous thereon, and let d_n denote the uniform distance from $g(t)$ to the span of $\{f_r(t); r = 0, \ldots, n\}$ in (a_1, b_1). Then for any number δ, $0 < \delta < \alpha$, there are numbers D (independent of n and g) and γ (independent of n), such that $d_n \leqslant D\|g\|_{L_2(R)}\exp(-\delta\rho^n)$.*

We shall use the following notation: By $\Sigma^{(k)}$ and $\Pi^{(k)}$ we shall denote sums and products of the form indicated, kth term deleted. For the theory of entire functions we shall refer to the book by R. P. Boas, Jr. [**8**].

PROOF OF LEMMA. We shall only consider the case in which $c_n \neq 0$ for all n, the other case being similar. Let $r_k(z) = \Pi^{(k)}(1 - z^2/c_n^2)$, and $\mu > 0$. As in the proof of [**5**, p. 33, Lemma 7.2] (with $\lambda_n = c_n^2$), we see that the sequence $\{\exp[(\mu/4)c_k^2]r_k(c_k)\}$ is bounded away from zero, say

$$\exp\left[(\mu/4)c_k^2\right]|r_k(c_k)| \geqslant D > 0. \tag{2}$$

Clearly $r_k(z) = P_k(z)P_k(-z)$, where

$$P_k(z) = \prod^{(k)} E(z/c_n, 1) = \prod^{(k)} (1 - z/c_n)\exp(z/c_n).$$

If $n_k(r)$ denotes the number of elements in the sequence $\{c_n, n \neq k\}$ within the disk of radius r, and $n(r)$ is similarly defined for the whole sequence $\{c_n\}$, it is clear that $n_k(r) \leqslant n(r)$. In view of this inequality, setting $|z| = r$ and applying to $P_k(z)$ the same technique employed in the proof of [**8**, pp. 29–30, 2.10.13], we readily see there is a function $u(r)$ (the same for all k), such that $\lim_{r\to\infty} u(r) = 0$, and

$$|r_k(z)| \leqslant \exp\left[u(r)r^2\right] \tag{3}$$

for all complex z. Setting

$$q_k(z) = q_k(\mu, z) = (2\pi)^{-1/2}\exp\left[(-\mu/4)(z^2 - c_k^2)\right]r_k(z)/r_k(c_k),$$

we see that

$$q_k(-c_n) = (2\pi)^{-1/2}\delta_{kn}. \tag{4}$$

In view of (2) and (3), a straightforward computation shows that

$$\int_R |q_k(x + yi)|^2\,dx \leqslant d_1^2\exp\left[\mu(y^2 + c_k^2)\right], \quad \text{and}$$

$$\int_R |(x + yi)q_k(x + yi)|^2\,dx \leqslant d_2^2\exp\left[\mu(y^2 + c_k^2)\right],$$

where d_1 and d_2 are independent of k (they are, of course, dependent on μ). Proceeding as in the proof of the necessity part of Theorem 3 in [**1**, pp. 304–305], we conclude that $q_k(z)$ is the Fourier transform of a function $h_k(t) = h_k(t, \mu)$ (i.e. $q_k(z) = (2\pi)^{-1/2}\int_R h_k(t)\exp(zti)\,dt$), such that $h_k(t)$ is continuous, and (for t real),

$$|h_k(t)| \leqslant d\exp\left[-t^2/(2\mu) + \mu c_k^2\right], \tag{5}$$

where d is independent of k. Let $\mu < 1/(2b)$. Then, if $m_k(t) = m_k(t, \mu) = h_k(t)/F(t)$, and bearing in mind that $h(t) = \exp(-bt^2)/F(t)$ is in $L_2(R)$ by hypothesis, it is clear from (5) that

$$|m_k(t)| \leqslant d\exp\left[-(1/(2\mu) - b)t^2 + \mu c_k^2\right]h(t). \tag{6}$$

Let $p_k(t)$ be the inverse Fourier transform of $m_k(t)$. By Plancherel's formula and (6), we see that

$$\int_R |p_k(t)|^2\,dt = \int_R |m_k(t)|^2\,dt \leqslant c^2\exp(2\mu c_k^2), \tag{7}$$

where c is independent of k.

From Plancherel's formula and (5), we also see that

$$\int_R p_k(t) f_n(t)\, dt = \int_R p_k(t) f(c_n - t)\, dt = \int_R m_k(t) F(t) \exp(-c_n t i)\, dt$$
$$= \int_R h_k(t) \exp(-c_n t i)\, dt = (2\pi)^{1/2} q_k(-c_n) = \delta_{kn}.$$

We have thus shown that

$$b_k(f_n) = \delta_{kn}. \tag{8}$$

Let $g(t)$ be a function in $L_2(R)$. Applying the Cauchy-Schwartz inequality and (7), we see that

$$|b_n(g) f(c_n - t)| \leqslant c \| g \|_{L_2(R)} \exp\left[\mu c_n^2 - \alpha(c_n - t)^2 \right].$$

Let δ be any number such that $0 < \delta < \alpha$. Setting $\mu = \alpha - \delta - \epsilon$, where $0 < \epsilon < \alpha - \delta$, we see that $\mu c_n^2 - \alpha(c_n - t)^2 = -\delta c_n - \epsilon(c_n - t)^2 + \gamma t^2$, whence we conclude that for this value of μ,

$$|b_n(g) f(c_n - t)| \leqslant c \| g \|_{L_2(R)} \exp(-\delta c_n^2 + \gamma t^2), \tag{9}$$

whence we readily conclude that

$$|S(g, t)| \leqslant M(t) \| g \|_{L_2(R)}, \tag{10}$$

where $M(t) = c \exp \gamma t^2 \sum \exp(-\delta c_r^2)$, and the conclusion follows from (6), (8), (9) and (10). Q.E.D.

Proof of Theorem 1. Assume that $g(t)$ is in the $L_2(R)$ closure of S. Let $\{g_n\}$ be a sequence of elements of S that converges to $g(t)$ in the $L_2(R)$ distance. Taking if necessary a subsequence thereof, we can assume without loss of generality that $\{g_n\}$ converges to $g(t)$ a.e. in R.

From (8) we readily conclude that $S(g_n, t) = g_n(t)$. Applying (10), we thus see that

$$|g_n(t) - S(g, t)| = |S(g_n, t) - S(g, t)|$$
$$= |S(g_n - g, t)| \leqslant M(t) \| g_n - g \|_{L_2(R)}. \tag{11}$$

Thus $S(g, t) = \lim_{n \to \infty} g_n(t)$, and therefore $g(t) = S(g, t)$, a.e., whence the conclusion follows. Q.E.D.

Proof of Theorem 2. Assume first that $|c| \neq |c_n|$ for all n. Since the sequence $\{c_n\}$ diverges, there is a number k such that $|c_k| < |c| < |c_{k+1}|$. Let $d_n = c_n$ if $n < k$, $d_k = c$, and $d_n = c_{n+1}$ if $n > k$. Clearly (1) is also satisfied (*with the same* ρ) by the sequence $\{d_n\}$. Let $r(z) = \prod(1 - z^2/c_n^2)$, and $P(z) = \prod^{(k)}(1 - z^2/d_n^2)$. Clearly, $r(c) = (1 - c^2/c_k^2)(1 - c^2/c_{k+1}^2)P(c)$. Let $\mu > 0$; inspection of the proof of [5, p. 33, Lemma 7.2] shows that

$$\exp\left[(\mu/4)c^2\right]|P(c)| = \exp\left[(\mu/4)c^2\right]|P(d_k)| \geqslant D > 0$$

(where D is independent of c), and therefore

$$|r(c)| \geqslant m_c^2 D. \tag{12}$$

Let $q(z) = q(\mu, z) = \exp[-(\mu/4)z^2]r(z)$, and $0 < \mu < 1/(2b)$. Proceeding again as in [**1**, pp. 304–305], we see that

$$q(z) = (2\pi)^{-1/2} \int_R m(t)F(t)\exp(zti)\,dt,$$

where $m(t) = m(\mu, t)$ is such that $|m(t)| \leqslant d\exp[-(1/(2\mu) - b)t^2]h(t)$, and $h(t) = \exp(-bt^2)/|F(t)|$ is in $L_2(R)$; thus the $L_2(R)$ norm of $m(t)$ is independent of c. Since $q(-c_n) = 0$, it readily follows from [**9**, p. 337, (V. 75)], that

$$|q(c)| = \left|\int_R m(t)F(t)\exp(cti)\,dt\right| \leqslant d_c\|m\|_{L_2(R)}.$$

Since $\mu < 1/(2b)$, and (12) implies that $|q(c)| \geqslant Dm_c^2\exp[-(\mu/4)c^2]$, the conclusion follows. If $|c| = |c_k|$ for some k, define $d_n = c_n$ if $n \neq k$, and $d_k = c\ (= -c_k)$. Thus if $r(z)$ is defined as above, $r(z) = (1 - z/c)\prod^{(k)}(1 + z/d_n)$, and therefore $r(c) = 2\prod^{(k)}(1 + d_k/d_n)$. Since the sequence $\{d_n\}$ satisfies (1), the conclusion follows as above. Q.E.D.

Proof of Theorem 3. Let $g(t)$ be a function in the $L_2(R)$ closure of S. From Theorem 1 we know that $g(t) = S(g, t)$ a.e. on R. However, it is readily seen from (9) and the continuity of the functions $f_n(t)$, that $S(g, t)$ is continuous on R, and therefore identical with $g(t)$ on (a_1, b_1). Thus,

$$g(t) = \sum_{r=0}^{\infty} b_r(g)f_r(t) \tag{13}$$

thereon. From (9) and (1) we know that if t is in (a_1, b_1), and $\eta^2 = \sup\{a_1^2, b_1^2\}$, then

$$\begin{aligned} |b_r(g)f_r(t)| &\leqslant c\|g\|_{L_2(R)}\exp(\gamma n^2)\exp(-\delta c_r^2) \\ &\leqslant c\|g\|_{L_2(R)}\exp(\gamma\eta^2 + c_0^2)\exp(-\delta\rho^r). \end{aligned} \tag{14}$$

Combining (13) and (14) we have

$$\begin{aligned} d_n &\leqslant \left|g(t) - \sum_{r=0}^{n} b_r(g)f_r(t)\right| \leqslant \sum_{r=n+1}^{\infty} |b_r(g)f_r(t)| \\ &\leqslant Q\|g\|_{L_2(R)} \sum_{r=n+1}^{\infty} \exp(-\delta\rho^r) \\ &= Q\|g\|_{L_2(R)}\left[\exp(-\delta\rho^{n+1})/(1 - \exp(-\delta\rho))\right], \end{aligned}$$

whence the conclusion follows. Q.E.D.

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