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MAPPING SURFACES HARMONICALLY INTO E^n

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ABSTRACT. A Weierstrass representation is given for harmonic maps from simply connected surfaces into E^3. The main result implies that the normals to a complete, harmonically immersed surface in E^n cannot omit a neighborhood of an (unoriented) direction if the mean curvature vector never vanishes, and the map from given to induced conformal structure is quasiconformal. In particular, the closure of the Gauss map to the complete graph of a harmonic function must be a hemisphere if the mean curvature never vanishes, and vertical projection is quasiconformal.

In this paper we use methods familiar from the study of minimal surfaces to obtain information about surfaces mapped harmonically into E^n. We give a Weierstrass representation for harmonic maps into E^3 (see Lemmas 4 and 5), and establish a correspondence between minimal surfaces in E^n and families of harmonically immersed surfaces. The main result implies that the normals to a complete, harmonically immersed surface in E^n cannot omit a neighborhood of some (unoriented) direction if the mean curvature vector never vanishes, and the identity map from given to induced conformal structure is quasiconformal. In particular then, the closure of the Gauss map to the complete graph of a harmonic function over any part of the (x_1, x_2)-plane in E^3 must be a hemisphere if the mean curvature never vanishes, and vertical projection is quasiconformal. The paper closes with an application of Bernstein's lemma.

Given a map $X: S \to E^n$ on a surface S, we write $\mathrm{I} = dX \cdot dX$. If X is an immersion, I is a Riemannian metric. By R we denote a conformal structure (or Riemann surface) on S. To any choice of R we associate with I a quadratic differential $\Omega = \Omega(\mathrm{I}, R)$ and an R-conformal metric $\Gamma = \Gamma(\mathrm{I}, R)$ defined as follows. If $z = x + iy$ is a conformal parameter on R, and if $\mathrm{I} = Edx^2 + 2Fdx\,dy + Gdy^2$, then

$$4\Omega = (E - G - 2iF)dz^2, \qquad 2\Gamma = (E + G)dz\,d\bar{z}.$$

When X is an immersion, I determines a conformal structure R_{I} on S, and $\Omega = 0$ exactly where $R = R_{\mathrm{I}}$. A map $X: R \to E^n$ is harmonic if and only if the Laplacian $\Delta X \equiv 0$ on R. A harmonic X is minimal if and only if $\Omega(\mathrm{I}, R) \equiv 0$. A nonconstant, minimal X is called a generalized minimal surface (see **[9]**). But for any harmonic map, $\Omega = \Omega(\mathrm{I}, R)$ is holomorphic (see **[4]** or **[6]**).

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We assume throughout that D is a simply connected domain in the z-plane containing the point $z = 0$. We think of D as a Riemann surface, despite the preferred conformal parameter z.

LEMMA 1. *If X: $D \to E^n$ is harmonic with $X = (x_k)$, there are analytic functions ϕ_k on D so that*

$$x_k = \mathrm{Re} \int_0^z \phi_k \, dz + c_k, \qquad k = 1, 2, \ldots, n, \tag{1}$$

for constants c_k. Setting

$$h = \sum_1^n \phi_k^2 \quad \text{and} \quad |\phi|^2 = \sum_1^n |\phi_k|^2, \tag{2}$$

one has

$$4\Omega = 4\Omega(\mathrm{I}, D) = h\,dz^2 \quad \text{and} \quad 2\Gamma = 2\Gamma(\mathrm{I}, D) = |\phi|^2 \, dz \, d\bar{z}.$$

If X is an immersion, Γ is complete if I *is, and for a positive function $\mu \leqslant 1$ the intrinsic curvatures of* I *and Γ are related by*

$$K(\Gamma) \leqslant \mu K(\mathrm{I}) \leqslant 0. \tag{3}$$

If $h \equiv 0$ and X is nonconstant, X is a generalized minimal surface.

PROOF. Since X is harmonic on D,

$$\frac{\partial}{\partial \bar{z}} \left(\frac{\partial X}{\partial z} \right) \equiv 0.$$

Set $z = x + iy$. Then

$$\phi = (\phi_k) = 2\left(\frac{\partial x_k}{\partial z} \right) = 2 \frac{\partial X}{\partial z}$$

is analytic, and

$$\int_0^z \phi_k \, dz = u_k + i v_k$$

for conjugate harmonic functions u_k and v_k, so that $\partial v_k / \partial x = - \partial u_k / \partial y$. But

$$\phi_k = \frac{d(u_k + i v_k)}{dz} = \frac{\partial(u_k + i v_k)}{\partial x} = \frac{\partial u_k}{\partial x} - i \frac{\partial u_k}{\partial y} = \frac{2 \partial u_k}{\partial z}.$$

Thus u_k and x_k differ by at most a constant, and (1) follows. Computation gives

$$\mathrm{I} = \sum_1^n (\mathrm{Re}\, \phi_k)^2 dx^2 - 2 \sum_1^n \mathrm{Re}\, \phi_k \mathrm{Im}\, \phi_k dx\, dy + \sum_1^n (\mathrm{Im}\, \phi_k)^2 dy^2,$$

so that $4\Omega(\mathrm{I}, D) = h\,dz^2$ and $2\Gamma(\mathrm{I}, D) = |\phi|^2 dz\, d\bar{z}$ as claimed. Suppose now that X is an immersion. By Lemma 1 in [5], Γ must be complete if I is. For any choice of a unit normal vector field ν, one has an associated second fundamental form

$$\mathrm{II}(\nu) = L dx^2 + 2M dx\, dy + N dy^2$$

with $\Delta X \cdot \nu \equiv L + N \equiv 0$. Thus $\det \mathrm{II}(\nu) = -(L^2 + M^2) \leqslant 0$. It follows that $K(\mathrm{I}) \leqslant 0$ since

$$K(\mathrm{I}) = \sum_1^{n-2} \frac{\det \mathrm{II}(\nu_j)}{\det \mathrm{I}} \tag{4}$$

for any choice (ν_j) of $n-2$ mutually orthogonal unit normal vector fields. (See [**3**, p. 205].) By Lemma 8.6 in [7], $K(\Gamma) \leqslant \mu K(\mathrm{I})$ because, in the notation of that paper, $0 < \mu = K(\Gamma, \mathrm{I}) \leqslant H^2(\Gamma, \mathrm{I}) \equiv 1$.

Remark 1. If $K(\mathrm{I}) \equiv 0$ on a harmonically immersed surface in E^n, then (4) and $\det \mathrm{II}(\nu_j) \leqslant 0$ show that $\mathrm{II}(\nu_j) \equiv 0$ for all j, so $X(S)$ lies in a 2-plane.

Remark 2. If $X: D \to E^n$ is harmonic, then near any point in D where the holomorphic quadratic differential $\Omega = \Omega(\mathrm{I}, D) \neq 0$, there is a local conformal parameter w in terms of which $4\Omega = h\,dz^2 = dw^2$. (See [**2**, p. 103].) In fact, $h = (dw/dz)^2$ for any such w. Thus w may be obtained as

$$w = \int_{z_0}^{z} \sqrt{h}\; dz$$

in some neighborhood of any z_0 where $\Omega \neq 0$. For arguments based on the use of such a parameter w, see [**6**] and [7].

Example 1. Any harmonic immersion $X: D \to E^n$ generates a family of "conjugate" harmonic immersions $X_\theta: D \to E^n$ defined by setting $\phi_\theta = e^{i\theta}\phi$ in (1) for any $0 \leqslant \theta \leqslant \pi$, so that $X_0 = X$. This construction is well known if X is minimal, in which case the conjugate surfaces are isometric. (See [**8**, p. 118].) If X is nonminimal, the first fundamental forms I_θ on conjugate harmonic surfaces will vary, but the associated conformal metrics $\Gamma(\mathrm{I}_\theta, D) = \Gamma(\mathrm{I}, D)$ remain the same! One easily checks that $\mathrm{I}_{\pi/2} = 2\Gamma - \mathrm{I}$. In the notation of [7], $\mathrm{I}_{\pi/2} = T(\mathrm{I}, D)$.

Example 2. Suppose $X: D \to E^n$ is harmonic, so that Lemma 1 applies. If $\tilde{\phi}_n = (\phi_n^2 - h)^{1/2}$ is analytic on D, the map $\tilde{X}: D \to E^n$ given by $\tilde{x}_k = x_k$ for $k \neq n$ and

$$\tilde{x}_n = \int_0^z \tilde{\phi}_n\, dz + c_n$$

is generalized minimal surface unless it is constant. Thus one can often associate a minimal $\tilde{X}$ to a harmonic X. (Indeed, this is always possible locally, wherever $\phi_n^2 \neq h$.) Conversely, suppose $X: D \to E^n$ is a generalized minimal surface, so that Lemma 1 applies again. One can *always* associate to X a family $\tilde{X}_u: D \to E^n$ of harmonic maps as follows. Let u be *any* fixed harmonic function on D. Set $\tilde{x}_k = x_k$ for $k \neq n$, and let $\tilde{x}_n = u$. (Thus $X = \tilde{X}_u$ when $u \equiv x_n$.) These constructions yield correspondences by "vertical" projection which are conformal with respect to the complex structure on D.

We denote by $\tilde{R}$ the universal cover of a Riemann surface R. We lift objects on R to $\tilde{R}$ with no change in notation. If $X: R \to E^n$ is a nonconstant, harmonic map, neither R nor $\tilde{R}$ can be compact, and $\tilde{R}$ is conformally equivalent to either the unit disc or the finite plane. (See [**1**].)

LEMMA 2. *If X: $R \to E^n$ is a complete, harmonic immersion, and if each ϕ_k given by Lemma* 1 *for X: $\tilde{R} \to E^n$ is never zero, then R is parabolic.*

PROOF. Since I is complete on $\tilde{R}$, so is $2\Gamma = |\phi|^2 dz\, d\bar{z}$. Since $|\phi|^2 = \Sigma_1^n |\phi_k|^2$ and no ϕ_k vanishes, at least one of the Riemannian metrics $|\phi_k|^2 dz\, d\bar{z}$ must be complete as well. But $\ln|\phi_k|^2$ is a harmonic function on $\tilde{R}$. By Lemma 9.2 in [9], $\tilde{R}$ is the plane, and R must be parabolic. (See [1].)

LEMMA 3. *If X: $R \to E^n$ is a complete, harmonic immersion, and if ϕ given by Lemma* 1 *for the immersion X: $\tilde{R} \to E^n$ satisfies*

$$\frac{|\langle \phi, b\rangle|^2}{|\phi|^2} \geqslant \varepsilon > 0 \tag{5}$$

for some constant ε and a unit vector b in E^n, then $X(R)$ must be a 2*-plane.*

REMARK 3. Condition (5) forces normals to avoid some neighborhood of the unoriented direction determined by b. Generally the converse fails.

PROOF. Because I is complete on $\tilde{R}$, so is $2\Gamma = |\phi|^2 dz\, d\bar{z}$. Since $\langle \phi, b\rangle = \Sigma_1^n b_k \phi_k$ is analytic on $\tilde{R}$, (5) shows that $\ln|\phi|^2$ is less than a harmonic function $\ln|\langle \phi, b\rangle|^2/\varepsilon$ on $\tilde{R}$. By Lemma 9.2 in [9], $\tilde{R}$ is the plane. But now, (5) shows that

$$\left|\frac{\phi_k}{\langle \phi, b\rangle}\right|^2 \leqslant \frac{|\phi|^2}{|\langle \phi, b\rangle|^2} \leqslant \frac{1}{\varepsilon}.$$

By Liouville's theorem, each of the entire functions $\phi_k/\langle \phi, b\rangle$ must be constant. It follows that

$$\frac{K(2\Gamma)}{4} = \frac{\langle \phi, d\phi/dz\rangle^2 - |\phi|^2 |d\phi/dz|^2}{|\phi|^6} \equiv 0.$$

(See [9, p. 126].) Thus (3) gives $K(\mathrm{I}) \equiv 0$, and by Remark 1, $X(R)$ must be a 2-plane.

Remark 3 suggests the following extension of a well-known result. (See [9, p. 120] or [6, p. 121].)

THEOREM. *If $\Omega(\mathrm{I}, R)$ has no isolated zeros for a complete, harmonic immersion X: $R \to E^n$, then $X(R)$ must be a* 2*-plane if the normals omit a neighborhood of some* (*unoriented*) *direction, and if* id: $R \to R_{\mathrm{I}}$ *is quasiconformal.*

PROOF. If $\Omega = \Omega(\mathrm{I}, R) \equiv 0$, X is minimal and the result is known. Otherwise, Ω never vanishes. Apply Lemma 1 to $\tilde{X}$: $\tilde{R} \to E^n$. Then

$$w = \int_0^z \sqrt{h}\; dz \tag{6}$$

provides an immersion of $\tilde{R}$ into the w-plane, since $\Omega = h dz^2$ never vanishes and $\tilde{R}$ is simply connected. Thus one can use w as a *local* conformal parameter anywhere on $\tilde{R}$. Lemma 1 then provides (by local application) an analytic $\tilde{\phi} = (\tilde{\phi}_k) = X_w$ defined *everywhere* on $\tilde{R}$ with $\Sigma_1^n \tilde{\phi}_k^2 \equiv 1$ and $2\Gamma(\mathrm{I}, \tilde{R}) = |\tilde{\phi}|^2\, dw\, d\bar{w}$. Thus $\mathrm{I} = |\mathrm{Re}\, \tilde{\phi}|^2\, du^2 + (|\mathrm{Re}\, \tilde{\phi}|^2 - 1)\, dv^2$. By quasiconformality, $|\mathrm{Re}\,\tilde{\phi}|^2 > 1 + \gamma$ for a fixed $\gamma > 0$. If b is a unit vector in the direction omitted by normals, either $\langle \mathrm{Re}\,\tilde{\phi}, b\rangle^2 \geqslant |\mathrm{Re}\,\tilde{\phi}|^2\delta$ or $\langle \mathrm{Im}\,\tilde{\phi}, b\rangle^2 \geqslant |\mathrm{Im}\,\tilde{\phi}|^2\delta = (|\mathrm{Re}\,\tilde{\phi}|^2 - 1)\delta$ for a fixed $\delta > 0$. Either way, (5) holds for $\phi = \tilde{\phi}$ and a fixed $\varepsilon > 0$. Because $\langle\tilde{\phi}, b\rangle$ is holomorphic, (5) shows that $\log|\tilde{\phi}|^2$ is less than a harmonic function on $\tilde{R}$. Since

$$2\Gamma(\mathrm{I}, \tilde{R}) = |\tilde{\phi}|^2\, dw\, d\bar{w} = |\tilde{\phi}|^2|h|^2\, dz\, d\bar{z}, \tag{7}$$

it follows that $\log|\tilde{\phi}h|^2$ is also majorized by a harmonic function on $\tilde{R}$. Because I is complete, so is $\Gamma(\mathrm{I}, \tilde{R})$. By Lemma 9.2 in **[9]**, $\tilde{R}$ is the plane, so that (6) must be a conformal equivalence with the whole w-plane. Lemma 3 now applies to the induced harmonic immersion of the w-plane.

Remark 4. Whenever $\Omega(\mathrm{I}, R) = 0$ for a harmonic immersion, $R = R_{\mathrm{I}}$ and the mean curvature vector $\mathcal{H}$ must vanish. (See **[6]**.) Thus our theorem shows that the normals to a complete, harmonically immersed surface in E^n cannot omit some neighborhood of an unoriented direction if $\mathcal{H}$ never vanishes and id: $R \to R_{\mathrm{I}}$ is quasiconformal.

The next result prepares the way for a Weierstrass representation of harmonic maps from simply connected domains into E^3. It generalizes Lemma 8.1 in **[9]**, where the symbol g is used for our $\sqrt{g}$.

Lemma 4. *Let $\mathfrak{D}$ be any domain in the z-plane. Suppose g is meromorphic on $\mathfrak{D}$, with f, h, fg and $(f^2g + h)^{1/2}$ analytic in $\mathfrak{D}$. Then the analytic functions*

$$\phi_1 = \tfrac{1}{2}f(1 - g), \quad \phi_2 = \tfrac{1}{2}if(1 + g), \quad \phi_3 = (f^2g + h)^{1/2} \tag{8}$$

satisfy

$$h = \sum_1^3 \phi_k^2. \tag{9}$$

Conversely, every triple of analytic functions ϕ_k on $\mathfrak{D}$ can be represented in the form (8) *with h given by* (9), *unless $\phi_1 \equiv i\phi_2$.*

Proof. The ϕ_k given by (8) clearly satisfy (9). Conversely suppose analytic functions ϕ_1, ϕ_2 and ϕ_3 are given on $\mathfrak{D}$ with $\phi_1 \not\equiv i\phi_2$. Define h on $\mathfrak{D}$ by (9) and set

$$f = \phi_1 - i\phi_2, \qquad g = \frac{\phi_3^2 - h}{(\phi_1 - i\phi_2)^2}. \tag{10}$$

By (9), one has $(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) = h - \phi_3^2$, so that

$$\phi_1 + i\phi_2 = \frac{h - \phi_3^2}{\phi_1 - i\phi_2} = -gf.$$

Thus (8) holds, with g meromorphic, and f, h, gf and $(f^2g + h)^{1/2}$ analytic, as claimed. (When $h \equiv 0$, $\sqrt{g}$ is meromorphic.)

The next result generalizes Lemma 8.2 in **[9]** and provides an extension of the classical Weierstrass representation.

LEMMA 5. *Suppose X: $D \to E^3$ is harmonic, so that Lemma* 1 *applies with $n = 3$. If $\phi_1 \equiv i\phi_2 \equiv 0$, x_1 and x_2 are constant so that $X(D)$ lies along a vertical line. If $\phi_1 \equiv i\phi_2 \not\equiv 0$, then $x_i - ix_2$ is analytic, and away from isolated zeros of ϕ_1, $x_1 - ix_2$ is a conformal parameter, so that $X(D)$ can be locally represented as the graph of a harmonic function $x_3 = x_3(x_1, x_2)$. If $\phi_1 \not\equiv i\phi_2$, the representation of the ϕ_k by* (8) *applies. The map X is then an immersion if and only if g has a pole of order m at every mth order zero of f, while $|g| \neq 1$ wherever $\bar{g}(g + h/f^2) \leqslant 0$.*

PROOF. Lemmas 1 and 4 establish all claims but the last. For X to be regular, one needs

$$X_x \times X_y = \operatorname{Im}\{\phi_2\bar{\phi}_3, \phi_3\bar{\phi}_1, \phi_1\bar{\phi}_2\} \neq 0.$$

If $\phi_1 \not\equiv i\phi_2$, (8) and (9) apply. Set $F = \pm (g + h/f^2)^{1/2}$. Then

$$X_x \times X_y = \tfrac{1}{2}|f|^2\left\{\operatorname{Re}(1 + \bar{g})F, \operatorname{Im}(1 - \bar{g})F, \tfrac{1}{2}(|g|^2 - 1)\right\},$$

so that

$$|X_x \times X_y|^2 = \tfrac{1}{4}|f|^4\left\{(1 + |g|^2)|F|^2 + 2\operatorname{Re}\bar{g}F^2 + \left(\tfrac{1}{2}(|g|^2 - 1)\right)^2\right\}.$$

Thus $|X_x \times X_y| \neq 0$ means that g has a pole of order m at every mth order zero of f. (Because fg is analytic, the order of the pole cannot exceed the order of the zero in any case.) Where $f \neq 0$, the condition $|X_x \times X_y| \neq 0$ forces $|g| \neq 1$ wherever $\bar{g}F^2 \leqslant 0$.

REMARK 5. If one sets $n = 3$ in Example 1, and uses the representation (8) for the harmonic map X: $D \to E^3$, the conjugate maps X_θ: $D \to E^3$ are represented by functions $f_\theta = e^{i\theta}f$, $g_\theta = g$ and $h_\theta = e^{2i\theta}h$.

REMARK 6. If one sets $n = 3$ in Example 2, then one needs $\sqrt{g}$ meromorphic in order to obtain a well-defined minimal map $\tilde{X}$ associated with a harmonic map X. Note that $\sqrt{g}$ then equals the composition of the Gauss spherical image map with sterographic projection (see [**9**, p. 66]).

COROLLARY TO LEMMA 5. *Suppose X: $D \to E^3$ is a harmonic immersion with* 1-1 *orthogonal projection onto the x_1x_2-plane. If $X(D)$ is not the graph of a harmonic function of x_1 and x_2, the function g from the representation* (8) *either has a zero of odd order, or is constant.*

PROOF. If $\phi_1 \equiv i\phi_2$ in Lemma 1, the first alternative indicated is valid. If $\phi_1 \not\equiv i\phi_2$, the representation (8) can be used. If $\sqrt{g}$ is not meromorphic, g has at least one zero of odd order. If $\sqrt{g}$ is meromorphic, the associated minimal surface $\tilde{X}$ also has 1-1 projection onto the (x_1, x_2)-plane. (See Remark 6). By Bernstein's theorem, $\tilde{X}(D)$ must be a plane. Thus the Gauss spherical image map of $\tilde{X}$ is constant, and so is g.

ADDED IN PROOF. If $\Omega(\mathrm{I}, R) \neq 0$ for a harmonic immersion X: $R \to E^n$, then (6) gives $w = u + iv$, and a vector field ν on $\tilde{R}$ in the direction $\partial/\partial u$ which maps R to the unit sphere in E^n. When I is complete, the proof of the theorem shows that $X(R)$ is a 2-plane if ν avoids a neighborhood of some "equator". (No quasiconformality assumption is required here.)

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