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RADII OF IMMERSED MANIFOLDS AND NONEXISTENCE OF IMMERSIONS

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ABSTRACT. Let M be a compact Riemannian manifold isometrically immersed in a complete Riemannian manifold N. By the radius of M in N, we mean the minimum of radii of closed geodesic balls in N which contain M. Using the concept of a radius, we will give a theorem about the nonexistence of isometric immersions, which is a generalization of J. D. Moore's result.

1. Introduction. Let M and N denote C^∞ Riemannian manifolds, K and K^* their respective sectional curvature functions. J. D. Moore [3] proved that when N is a complete simply connected Riemannian manifold with $a \leqslant K^* \leqslant b \leqslant 0$, and M is a compact Riemannian manifold with $K \leqslant a - b$, M possesses no isometric immersion in N, unless $\dim N \geqslant 2 \dim M$. On the other hand H. Jacobowitz [1] showed that an isometric immersion of an n-dimensional compact Riemannian manifold with sectional curvature always less than λ^{-2} into Euclidean space of dimension $2n - 1$ can never be contained in a ball of radius λ. In this note, using methods similar to those of [3], we generalize the results of J. D. Moore and H. Jacobowitz. At first we define a positive continuous function $C(b, d)$ on $(-\infty, 0] \times (0, \infty)$ by

$$C(b, d) = \begin{cases} \dfrac{1}{d} & \text{if } b = 0, \\ \sqrt{-b}\, \coth (d\sqrt{-b}\,) & \text{if } b < 0. \end{cases}$$

This function is monotonically decreasing with respect to b and also d. We will prove

THEOREM 1. *Let N be a complete simply connected Riemannian manifold whose sectional curvatures $K^*(\sigma)$ satisfy the inequalities*

$$a \leqslant K^*(\sigma) \leqslant b \leqslant 0, \tag{1}$$

M be a compact Riemannian manifold with diameter d. Assume that at every point of M, there is a p-dimensional subspace in the tangent space, along whose plane elements σ, it holds

$$K(\sigma) \leqslant a + C^2(b, d). \tag{2}$$

If $\dim N < \dim M + p$, *then M cannot be isometrically immersed in N.*

As the function $C(b, d)$ satisfies $C^2(b, d) > -b$, this theorem strengthens the

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result in [3]. Theorem 1 is immediate from the following theorem which generalizes H. Jacobowitz's result [1].

THEOREM 2. *Assume that N satisfies the same conditions as in Theorem* 1. *Let d be some positive constant. Let M be a compact manifold such that at every point of M, there is a p-dimensional subspace in the tangent space, along whose plane elements σ, the inequality* (2) *holds. If* $\dim N < \dim M + p$, *then no isometric immersion of M into N is contained in a ball of radius d.*

2. Radii of immersed manifolds. We will deal with a ball containing an immersed manifold as in [1]. Let M be a compact Riemannian manifold isometrically immersed in a complete Riemannian manifold N. Let $d(\, , \,)$ be the distance function of N. For any point $x \in N$ and any $r > 0$, put $B(x, r) = \{y \in N, d(x, y) < r\}$. Then we set

$$r(M) = \inf\{r; M \subset B(x, r)\} = \inf\{\max\{d(x, y), y \in M\}, x \in N\}. \qquad (3)$$

As M is compact, we can prove there is a point $x_0 \in N$ such that $B(x_0, r(M)) \supset M$. Moreover there is a point $y_0 \in M$ such that $d(x_0, y_0) = r(M)$. We will call $r(M)$ *the radius of M in N* and $B(x_0, r(M))$ *a minimum ball containing M*. Generally, there are several minimal balls containing M. For example, let $S^1 = \{(x_1, x_2); x_1^2 + x_2^2 = 1\}$ be naturally imbedded in $S^2 = \{(x_1, x_2, x_3); x_1^2 + x_2^2 + x_3^2 = 1\}$. Then minimal balls containing S^1 are $B((0, 0, 1), \pi/2)$ and $B((0, 0, -1), \pi/2)$. But there is only one minimal ball for a compact manifold immersed in a euclidean space. In fact we have

THEOREM 3. *Let M be a compact manifold immersed in an n-dimensional euclidean space E^n. Then there is only one point $x_0 \in E^n$ such that* $r(M) = \max\{d(x_0, y), y \in M\}$.

PROOF. Take a point x_0 which satisfies the above equality. Let $S(x_0, r(M)) = \{y \in M, d(x_0, y) = r(M)\}$. At first we will prove that $S(x_0, r(M))$ contains more than one point. Suppose there is only one point y_0 in M with $d(x_0, y_0) = r(M)$. Take a positive δ satisfying $\delta < r(M)/2$. Put $r_1 = \max\{d(x_0, y), M - B(y_0, \delta) \ni y\}$. Then $r_1 < r(M)$. Hence if $x_1 \in B(x_0, \varepsilon_1/2)$,

$$\max\{d(x_1, y), M - B(y_0, \delta) \ni y\} < r_1 + \frac{\varepsilon_1}{2} < r(M),$$

where we put $\varepsilon_1 = r(M) - r_1$. Let $x_\varepsilon = \varepsilon x_0 + (1 - \varepsilon) y_0$. An easy calculation shows that for any $y \in M \cap B(y_0, \delta)$,

$$d(x_\varepsilon, y) < r(M)^2 - 2\varepsilon\delta + \varepsilon^2 < r(M)^2 \qquad (0 < \varepsilon < 2\delta).$$

Thus if $\varepsilon < \min(\varepsilon, 2\delta)$, $\max\{d(x_\varepsilon, y), y \in M\} < r(M)$. This is contrary to the definition of $r(M)$. Let $\Pi(x_0)$ be the k-dimensional affine subspace of E^n spanned by $S(x_0, r(M))$, where $1 \leqslant k \leqslant n$. Take $k + 1$ points $y_0, y_1, \ldots, y_k$ from $S(x_0, r(M))$ such that $y_1 - y_0, y_2 - y_0, \ldots, y_k - y_0$ are linearly independent. Then they obviously span $\Pi(x_0)$. If $k = n$, there is no point except x_0 whose distances from $y_0, y_1, \ldots, y_n$ are $r(M)$. Hence let $1 \leqslant k < n$. We will prove $x_0 \in \Pi(x_0)$.

Suppose that $x_0 \notin \Pi(x_0)$. We may assume that $x_0 = 0$ and $\Pi(x_0) = \{(a, x_2, \ldots, x_{k+1}, 0, \ldots, 0)\}$, where a is a positive constant. Set $I = \{(x_1, \ldots, x_n), \frac{3}{4} a < x_1\}$. Then $S(x_0, r(M)) \subset I$ and $M - I$ is compact. Hence we have $\max\{d(x_0, y), y \in M - I\} = r_1 < r(M)$. Let $x_\varepsilon = (\varepsilon, 0, \ldots, 0) \in R^n$ $(0 < \varepsilon < a)$. Then we can prove

$$\max\{d(x_\varepsilon, y), y \in M \cap I\} < \sqrt{r(M)^2 - \frac{\varepsilon^2}{2}} < r(M).$$

Let $0 < \varepsilon < \min((r(M) - r_1)/2, a)$. It follows that

$$\max\{d(x_\varepsilon, y), y \in M - I\} < r(M) - \frac{\varepsilon}{2} < r(M).$$

Thus we obtain $\max\{d(x_\varepsilon, y), y \in M\} < r(M)$. But this is contrary to our assumption. Now, $x_0 \in \Pi(x_0)$. Then there is only one point on $\Pi(x_0)$ whose distances from $y_0, \ldots, y_k$ are $r(M)$ as similarly as in the case $k = n$. Lastly we assume that there is a point $x_1 \notin \Pi(x_0)$ with $B(x_1, r(M)) \supset M$. But this is impossible because it follows from this assumption that at least one of $d(x_1, y_i)$ $(i = 0, 1,, \ldots, k)$ should be greater than $r(M)$. The proof is now completed exactly.

3. Proofs of Theorems 1 and 2. Firstly we assume that M and N satisfy the hypotheses of Theorem 2. Moreover suppose that M is isometrically immersed in N and contained in a ball of radius d. Then we have $r(M) \leqslant d$. Take $x_0 \in N$ and $y_0 \in M$ satisfying $r(M) = d(x_0, y_0)$. Let γ: $[0, 1] \to N$ be a minimal geodesic with $\gamma(0) = x_0$, $\gamma(1) = y_0$. For each unit tangent vector $v \in T_{y_0}M$, there is a unique Jacobi field V along γ such that $V(0) = 0$, $V(1) = v$. Corresponding to V, we have a one-parameter family of geodesics from x_0 to M, $\gamma_s(t) = \gamma(s, t)$: $(-\varepsilon, \varepsilon) \times [0, 1]$, which satisfies $\gamma_0(t) = \gamma(t)$, $(\partial\gamma(s, t)/\partial s)(0, t) = V(t)$. We set

$$L(\gamma_s) = \frac{1}{2}\int_0^1 \langle \gamma_s', \gamma_s' \rangle \, dt.$$

Then from the definition of γ, it follows that $L(\gamma_s) \leqslant L(\gamma)$. Hence

$$0 \geqslant \left(\frac{d^2}{ds^2} L(\gamma_s)\right)_{s=0} = I(V, V) + \langle \alpha(v, v), \gamma'(1) \rangle, \tag{4}$$

where α is the second fundamental form of M in N, and $I(\, , \,)$ is the index form. Taking a proper Jacobi field on a space of constant curvature b, J. D. Moore **[2]** proved that $I(V, V) \geqslant r(M)C(b, r(M))$. Hence from (4) we get $\langle \alpha(v, v), \gamma'(1) \rangle < -r(M)C(b, r(M))$. Since $\|\gamma'(1)\| = r(M)$, we obtain for all unit vectors $v \in T_{y_0}M$, $\|\alpha(v, v)\| > C(b, r(M))$. On the other hand, if σ is a plane element which is spanned by v, w and satisfies (2), it holds that

$$\langle \alpha(v, v), \alpha(w, w) \rangle - \|\alpha(v, v)\|^2 = K(\sigma) - K^*(\sigma) \leqslant C^2(b, d).$$

Since the function $C(b, d)$ is monotonically decreasing, it follows that $C(b, d) \leqslant C(b, r(M))$. Thus the proof of Theorem 1 is finished by the following lemma which was proved essentially by T. Otsuki **[4]**. We will prove by an argument due to T. A. Springer **[2**, Chapter 8, §4].

LEMMA. *Let* $\alpha: R^p \times R^p \to R^m$ *be a symmetric bilinear mapping and* $\langle\ ,\ \rangle$ *be a positive definite inner product on* R^m. *If there is a nonnegative constant C which satisfies*

$$\langle \alpha(v, v), \alpha(w, w)\rangle - \langle \alpha(v, w), \alpha(v, w)\rangle \leqslant C^2, \quad \langle \alpha(v, v), \alpha(v, v)\rangle > C^2,$$

or

$$\langle \alpha(v, v), \alpha(w, w)\rangle - \langle \alpha(v, w), \alpha(v, w)\rangle < C^2,$$
$$\langle \alpha(v, v), \alpha(v, v)\rangle \geqslant C^2, \qquad \alpha(v, v) \neq 0,$$

for all nonzero $v, w \in R^p$, *then we have* $m \geqslant p$.

PROOF. We extend α to a symmetric complex bilinear mapping of $C^p \times C^p \to C^m$. The equation $\alpha(z, z) = 0$ is equivalent to a system of m quadratic equations. If $m < p$, then this system of m equations has a nonzero solution $z = x + \sqrt{-1}\ y$, where $x, y \in R^p, y \neq 0$. As $\alpha(y, y) \neq 0$, from $\alpha(z, z) = 0$, we get $\alpha(x, x) = \alpha(y, y) \neq 0$ and $\alpha(x, y) = 0$. This is contrary to our assumption.

Theorem 1 follows easily from Theorem 2. Let M be a compact Riemannian manifold with diameter d. If M is isometrically immersed in N, it should be contained in some ball of radius d. Thus we have Theorem 1.

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