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ON RECURRENCE OF A RANDOM WALK IN THE PLANE

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ABSTRACT. The purpose of this note is to establish a sufficient condition for recurrence of a random walk (S_n) in R^2. It follows from it that if $S_n/n^{1/2}$ is asymptotically normal then we have recurrence.

Let $X_1, X_2, \ldots$ be independent, identically distributed random variables in R^k, $k \geqslant 1$, with common distribution F, and let for $n \geqslant 1$, $S_n = \sum_1^n X_i$, $S_0 = 0$. The random walk $S = (S_n)_0^\infty$ has a point of recurrence at x if, for every $\varepsilon > 0$,

$$P(|S_n - x| < \varepsilon \text{ i.o.}) = 1. \tag{1}$$

It is well known that the set of recurrence points is either empty or equals the smallest closed additive group containing the support of F, see **[1]** or **[3**, §8.3]. In the latter case we say that S is recurrent. Also well known is the following criterion: S is recurrent if and only if

$$\sum_0^\infty P(|S_n| \leqslant \varepsilon) = \infty \tag{2}$$

for some $\varepsilon > 0$, see the references above, and (2) holds if and only if

$$\overline{\lim_{r \nearrow 1}} \int_A \operatorname{Re} \frac{1}{1 - rf(t)}\, dt = \infty \tag{3}$$

for all neighborhoods A of 0, where f is the characteristic function of F: the criterion (3) is Theorem 3 in **[1]**.

The study of recurrence has been carried out, to a large extent, by using (3) and related criteria. For example, in one dimension $E[X_i] = 0$ implies recurrence, which is rather easily deduced from (3). In **[2]** a probabilistic (combinatorial) proof is given that the weaker assumption $S_n/n \xrightarrow{P} 0$ is sufficient for recurrence. In that paper it is also claimed that if, in R^2, $S_n/n^{1/2}$ is asymptotically normal, which is the case when $E[X_i] = 0$ (zero vector) and $E[|X_i|^2] < \infty$, then we have recurrence. However, the argument indicated for this result in **[2]**, and also in **[3**, Problem 14, p. 274], is misleading to say the least. This was discovered by students in Chung's class in 1977 and was first corrected by him then: it is the main purpose of this note to settle this matter.

For $y = (y_1, \ldots, y_k) \in R^k$, $k \geqslant 2$, we let $|y| = \max_{1 \leqslant i \leqslant k} |y_i|$ throughout.

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PROPOSITION. *Consider a random walk $S = (S_n)_0^\infty$ in R^2. If there exists an increasing, continuous function $h \geqslant 0$ on some interval* $[0, c]$ *such that $\int_0^c (h(u)/u)du = \infty$ and*

$$\varliminf_{n\to\infty} P(|S_n| \leqslant x \cdot n^{1/2}) \geqslant x^2 \cdot h(x) \tag{4}$$

for each $x \in [0, c]$, then S is recurrent.

PROOF. Since $x^2 \cdot h(x)$ is uniformly continuous on $[0, c]$, and since the functions $\inf_{n \geqslant k} P(|S_n| \leqslant x \cdot n^{1/2})$ increase with x, the inequality in (4) holds uniformly in x in the sense that for each $\varepsilon > 0$ there exist $n(\varepsilon)$ such that

$$\inf_{n \geqslant n(\varepsilon)} P(|S_n| \leqslant x \cdot n^{1/2}) \geqslant x^2 \cdot h(x) - \varepsilon \tag{5}$$

for all $x \in [0, c]$. Furthermore, for integers $m \geqslant 1$ we have

$$\sum_0^\infty P(|S_n| \leqslant m) \leqslant 4m^2 \cdot \sum_0^\infty P(|S_n| \leqslant 1).$$

This inequality was proved in [3, Lemma 1, p. 268] for R^1 with the constant $2m$ on the right side. The same argument yields the result for R^2 with the constant $(2m)^2$ due to our definition of S_n indicated above. By virtue of (2), it is hence sufficient to prove

$$\varliminf_{m\to\infty} m^{-2} \sum_0^\infty P(|S_n| \leqslant m) = \infty.$$

Fix $C > 0$ arbitrarily large. Take $B > 0$ so large that

$$\int_{B^{-1}}^c \frac{h(u)}{u}\, du \geqslant C.$$

Let $\varepsilon > 0$ be so small that

$$\inf_{n \geqslant n(\varepsilon)} P(|S_n| \leqslant x \cdot n^{1/2}) \geqslant x^2 \cdot h(x)/2$$

for $B^{-1} \leqslant x \leqslant c$, which is possible because of (5) and the monotonicity of $x^2 \cdot h(x)$. Now, if $n \geqslant n(\varepsilon)$ and $B^{-1} \leqslant m \cdot n^{-1/2} \leqslant c$, we have $P(|S_n| \leqslant m) \geqslant m^2 \cdot n^{-1} \cdot h(m \cdot n^{-1/2})/2$, so

$$\begin{aligned}
\varliminf_{m\to\infty} m^{-2} \sum_0^\infty P(|S_n| \leqslant m) &\geqslant \varliminf_{m\to\infty} m^{-2} \cdot \sum_{\substack{n \geqslant n(\varepsilon) \\ B^{-1} \leqslant m \cdot n^{-1/2} \leqslant c}} P(|S_n| \leqslant m) \\
&\geqslant \frac{1}{2} \varliminf_{m\to\infty} \int_{m^2 \cdot c^{-2}}^{B^2 m^2} \frac{h(m \cdot x^{-1/2})}{x}\, dx \\
&= \frac{1}{2} \int_{c^{-2}}^{B^2} \frac{h(x^{-1/2})}{x}\, dx = \int_{B^{-1}}^c \frac{h(u)}{u}\, du \geqslant C.
\end{aligned}$$

Since C is arbitrary, $\sum_0^\infty P(|S_n| \leqslant 1) = \infty$ and hence S is recurrent. □

If $S_n/n^{1/2}$ is asymptotically normal, the Proposition renders S recurrent: simply let $c = 1$ and let $h(x)$ be constant $= \min_{|y| \leqslant 1} g(y)$, where g is the relevant normal density.

It may be noticed that for every $\beta < 2$ there is a F such that $E[|X_i|^\beta] < \infty$ but S is transient. Namely, let $X_i = (X_i', X_i'')$ where the variables X_i', X_i'' are independent, stable and symmetric with index α, $\beta < \alpha < 2$: such a distribution has continuous density and all moments of order $< \alpha$ finite, see [4, Lemma 2, p. 545]. With $S_n' = \Sigma_1^n X_i'$, we obtain

$$\begin{aligned} P(|S_n| \leqslant 1) &= P(|S_n'| \leqslant 1)^2 = P\big(|S_n' \cdot n^{-1/\alpha}| \leqslant n^{-1/\alpha}\big)^2 \\ &= P\big(|X_i'| \leqslant n^{-1/\alpha}\big)^2 \leqslant 2 \cdot \gamma \cdot n^{-2/\alpha}, \end{aligned}$$

where γ is the supremum of the density of F on the interval $[-1, 1]$. Hence, $\Sigma_0^\infty P(|S_n| \leqslant 1)$ is finite, S is transient.

With a function h as in the Proposition, if $\underline{\lim}_{n\to\infty} P(|S_n| \leqslant x \cdot n) \geqslant x \cdot h(x)$ for each $x \in [0, c]$ for a one-dimensional random walk S, then S is recurrent. This sufficient condition covers the result by Chung and Ornstein, but also, for example, the case when F is a Cauchy distribution such that S_n/n is distributed like X_1: then $h(x)$ = a suitable constant will do.

Since h is allowed to tend to 0 as $x \searrow 0$, the question arises whether we can find a distribution with slightly heavier tails than that of the Cauchy distribution so that $P(|S_n| \leqslant x \cdot n) \to 0$ for all $x > 0$ as $n \to \infty$, and still have recurrence: it turns out that a symmetric distribution on the integers with $P(X_i = k) = \gamma \cdot \log(1 + |k|) \cdot (1 + k^2)^{-1}$, γ a normalizing constant, is such a distribution. To prove this, Fourier methods seem inevitable.

In order to illuminate that the condition $\int_0^c (h(u)/u)\,du = \infty$ is crucial, let $h \geqslant 0$ be increasing and such that $\int_0^c (h(u)/u)\,du < \infty$ for some $c > 0$ and suppose that

$$\overline{\lim_{n\to\infty}} \left[\sup_{0 \leqslant x \leqslant c} P\big(|S_n| \leqslant x \cdot n^{1/2}\big)/x^2 \cdot h(x) \right] < \infty$$

for a random walk S in R^2. Then S is transient, as is readily verified.

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