ON RECURRENCE OF A RANDOM WALK IN THE PLANE

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ABSTRACT. The purpose of this note is to establish a sufficient condition for recurrence of a random walk \((S_n)\) in \(\mathbb{R}^2\). It follows from it that if \(S_n/n^{1/2}\) is asymptotically normal then we have recurrence.

Let \(X_1, X_2, \ldots\) be independent, identically distributed random variables in \(\mathbb{R}^k, k > 1\), with common distribution \(F\), and let for \(n > 1\), \(S_n = \sum_1^n X_i, S_0 = 0\). The random walk \(S = (S_n)\) has a point of recurrence at \(x\) if, for every \(\varepsilon > 0\),

\[
P(|S_n - x| < \varepsilon \text{ i.o.}) = 1.
\]

It is well known that the set of recurrence points is either empty or equals the smallest closed additive group containing the support of \(F\), see [1] or [3, §8.3]. In the latter case we say that \(S\) is recurrent. Also well known is the following criterion: \(S\) is recurrent if and only if

\[
\sum_0^\infty P(|S_n| < \varepsilon) = \infty
\]

for some \(\varepsilon > 0\), see the references above, and (2) holds if and only if

\[
\lim_{r \to 1} \int_A \text{Re} \left\{ \frac{1}{1 - rf(t)} \right\} dt = \infty
\]

for all neighborhoods \(A\) of 0, where \(f\) is the characteristic function of \(F\): the criterion (3) is Theorem 3 in [1].

The study of recurrence has been carried out, to a large extent, by using (3) and related criteria. For example, in one dimension \(E[X_i] = 0\) implies recurrence, which is rather easily deduced from (3). In [2] a probabilistic (combinatorial) proof is given that the weaker assumption \(S_n/n \to 0\) is sufficient for recurrence. In that paper it is also claimed that if, in \(\mathbb{R}^2\), \(S_n/n^{1/2}\) is asymptotically normal, which is the case when \(E[X_i] = 0\) (zero vector) and \(E[|X|^2] < \infty\), then we have recurrence. However, the argument indicated for this result in [2], and also in [3, Problem 14, p. 274], is misleading to say the least. This was discovered by students in Chung’s class in 1977 and was first corrected by him then: it is the main purpose of this note to settle this matter.

For \(y = (y_1, \ldots, y_k) \in \mathbb{R}^k, k > 2\), we let \(|y| = \max_{1 \leq i \leq k} |y_i|\) throughout.
Proposition. Consider a random walk \( S = (S_n)_{n \geq 0} \) in \( \mathbb{R}^2 \). If there exists an increasing, continuous function \( h > 0 \) on some interval \([0, c]\) such that 
\[
\int_0^c (h(u)/u) \, du = \infty
\]
for each \( x \in [0, c] \), then \( S \) is recurrent.

Proof. Since \( x^2 \cdot h(x) \) is uniformly continuous on \([0, c]\), and since the functions 
\[
\inf_{n \geq n(e)} P(|S_n| < x \cdot n^{1/2}) \quad \text{increase with } x,
\]
the inequality in (4) holds uniformly in \( x \) in the sense that for each \( \epsilon > 0 \) there exist \( n(\epsilon) \) such that
\[
\inf_{n \geq n(\epsilon)} P(|S_n| < x \cdot n^{1/2}) > x^2 \cdot h(x) - \epsilon
\]
for all \( x \in [0, c] \). Furthermore, for integers \( m > 1 \) we have
\[
\sum_0^\infty P(|S_n| < m) < 4m^2 \cdot \sum_0^\infty P(|S_n| < 1).
\]
This inequality was proved in [3, Lemma 1, p. 268] for \( R^1 \) with the constant \( 2m \) on the right side. The same argument yields the result for \( R^2 \) with the constant \( (2m)^2 \) due to our definition of \( S_n \) indicated above. By virtue of (2), it is hence sufficient to prove
\[
\lim_{m \to \infty} m^{-2} \sum_0^\infty P(|S_n| < m) = \infty.
\]
Fix \( C > 0 \) arbitrarily large. Take \( B > 0 \) so large that
\[
\int_{B^{-1}}^c \frac{h(u)}{u} \, du > C.
\]
Let \( \epsilon > 0 \) be so small that
\[
\inf_{n \geq n(\epsilon)} P(|S_n| < x \cdot n^{1/2}) > \frac{x^2 \cdot h(x)}{2}
\]
for \( B^{-1} < x < c \), which is possible because of (5) and the monotonicity of \( x^2 \cdot h(x) \). Now, if \( n > n(\epsilon) \) and \( B^{-1} < m \cdot n^{-1/2} < c \), we have \( P(|S_n| < m) > m^2 \cdot n^{-1} \cdot h(m \cdot n^{-1/2})/2 \), so
\[
\lim_{m \to \infty} m^{-2} \sum_0^\infty P(|S_n| < m) > \lim_{m \to \infty} m^{-2} \cdot \sum_{n \geq n(\epsilon)} P(|S_n| < m)
\]
\[
> \frac{1}{2} \lim_{m \to \infty} \int_{m^2 \cdot c^{-2}}^{B^2 m^2} \frac{h(m \cdot x^{-1/2})}{x} \, dx
\]
\[
= \frac{1}{2} \int_{c^{-2}}^{B^2} \frac{h(x^{-1/2})}{x} \, dx = \int_{B^{-1}}^c \frac{h(u)}{u} \, du > C.
\]
Since \( C \) is arbitrary, \( \sum_0^\infty P(|S_n| < 1) = \infty \) and hence \( S \) is recurrent. \( \square \)

If \( S_n/n^{1/2} \) is asymptotically normal, the Proposition renders \( S \) recurrent: simply let \( c = 1 \) and let \( h(x) \) be constant \( = \min_{|y| < 1} g(y) \), where \( g \) is the relevant normal density.
It may be noticed that for every \( \beta < 2 \) there is a \( F \) such that \( E[|X_i|^{\beta}] < \infty \) but \( S \) is transient. Namely, let \( X_i = (X'_i, X''_i) \) where the variables \( X'_i, X''_i \) are independent, stable and symmetric with index \( \alpha, \beta < \alpha < 2 \); such a distribution has continuous density and all moments of order < \( \alpha \) finite, see [4, Lemma 2, p. 545]. With \( S'_n = \sum X'_i \), we obtain

\[
P(|S'_n| < 1) = P(|S'_n| < 1)^2 = P(|S'_n \cdot n^{\frac{1}{\alpha}}| < n^{-\frac{1}{\alpha}})^2 = P(|X'_i| < n^{-\frac{1}{\alpha}})^2 \leq 2 \cdot \gamma \cdot n^{-2/\alpha},
\]

where \( \gamma \) is the supremum of the density of \( F \) on the interval \([-1, 1]\). Hence, \( \Sigma_0^\infty P(|S'_n| < 1) \) is finite, \( S \) is transient.

With a function \( h \) as in the Proposition, if \( \lim_{n \to \infty} P(|S'_n| < x \cdot n) > x \cdot h(x) \) for each \( x \in [0, c] \) for a one-dimensional random walk \( S \), then \( S \) is recurrent. This sufficient condition covers the result by Chung and Ornstein, but also, for example, the case when \( F \) is a Cauchy distribution such that \( S'_n/n \) is distributed like \( X'_1 \); then \( h(x) = \) a suitable constant will do.

Since \( h \) is allowed to tend to 0 as \( x \to 0 \), the question arises whether we can find a distribution with slightly heavier tails than that of the Cauchy distribution so that \( P(|S'_n| < x \cdot n) \to 0 \) for all \( x > 0 \) as \( n \to \infty \), and still have recurrence: it turns out that a symmetric distribution on the integers with \( P(X_i = k) = \gamma \cdot \log(1 + |k|) \cdot (1 + k^2)^{-1}, \gamma \) a normalizing constant, is such a distribution. To prove this, Fourier methods seem inevitable.

In order to illuminate that the condition \( \int_0^{\infty} (h(u)/u) du = \infty \) is crucial, let \( h > 0 \) be increasing and such that \( \int_0^{\infty} (h(u)/u) du < \infty \) for some \( c > 0 \) and suppose that

\[
\lim_{n \to \infty} \sup_{0 < x < c} \frac{P(|S'_n| < x \cdot n^{1/2})}{x^2 \cdot h(x)} < \infty
\]

for a random walk \( S \) in \( \mathbb{R}^2 \). Then \( S \) is transient, as is readily verified.

REFERENCES


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