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HOMOTOPY AND UNIFORM HOMOTOPY. II

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ABSTRACT. An elementary proof of the Bounded Lifting Lemma is given, together with a proof that homotopy and uniform homotopy do not agree for maps into compact spaces with infinite fundamental groups even though they can agree for maps into a noncompact space with infinite fundamental group.

The purpose of this paper is two-fold: (1) To give a short, unified and a great deal more transparent proof of the main geometrical results of [1], upon which all of [1] and [2] depend. (2) To give a proof that if Y is compact and $\pi_1 Y$ is infinite then $[\beta -, Y]$ is not a homotopy functor. It follows that the result of [1] concerning the relation between homotopy and uniform homotopy for finite-dimensional normal spaces is best possible. We wish to thank J. Keesling for his observation with regard to (2).

A fibration $p: E \to B$, by which we will mean a (Hurewicz) fibration such that B has a numerable covering $\{U_\alpha\}$ with $p^{-1}(U_\alpha)$ trivial in the sense of Dold [4], is said to have the *bounded lifting property* (BLP) with respect to a subcategory $\mathfrak{T}$ of $\mathfrak{T}\mathfrak{O}\mathfrak{P}$, the category of topological spaces and maps, if for every space X in $\mathfrak{T}$ and map $f: X \to E$ such that pf is bounded there exist a bounded map $g: X \to E$ which is homotopic to f over p. (A bounded map is one for which the closure of the image is compact.) That is to say that any lift to E of a bounded map into B is homotopic over p to a bounded map. We say p has BLP($\mathfrak{T}$).

THEOREM 1 [1, (2.3) AND (3.3)]. *Let F be the fiber of $p: E \to B$; then* (1) *if F has the homotopy type of a compact space then p has BLP($\mathfrak{T}\mathfrak{O}\mathfrak{P}$),* (2) *if F has the homotopy type of a CW-complex of finite type* (*i.e. finitely many cells in each dimension*) *then p has BLP(fdNorm). Here fdNorm denotes the category of finite dimensional normal spaces.*

A space Y is said to have the *relative compressibility property* (RCP) with respect to $\mathfrak{T}$ if for any space X in $\mathfrak{T}$, subspace A of X and map $f: X \to Y$ such that $\overline{f(A)}$ is compact, there exists a homotopy $H: X \times I \to Y$ such that $H_0 = f$ and $\overline{H((X \times \{1\}) \cup (A \times I))}$ is compact. We say that Y has RCP($\mathfrak{T}$).

Clearly, a compact space has RCP($\mathfrak{T}\mathfrak{O}\mathfrak{P}$) and if Z has RCP($\mathfrak{T}$) and Z dominates Y (or in particular if Z is homotopically equivalent to Y) then Y has RCP($\mathfrak{T}$). So the theorem will be a consequence of the following two lemmas.

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LEMMA 1. *If T is closed under closed subspaces and F has RCP($\mathfrak{T}$) then p has BLP($\mathfrak{T}$).*

LEMMA 2. *A CW-complex of finite type has RCP(fdNorm).*

PROOF OF LEMMA 1. Let X be in $\mathfrak{T}$ and f: $X \to E$ a map such that $h = pf$ is bounded. By restricting to $\overline{h(X)}$ if necessary we may assume that B is compact. By our definition of fibration, there exists a finite open cover $\{U_i\}_{i=1}^n$ of B such that $p^{-1}(\overline{U}_i)$ is fiber homotopy equivalent to $\overline{U}_i \times F$. Let ϕ_i be such a homotopy equivalence and ψ_i its inverse.

Let $\{V_i\}$ be an open covering of B such that $\overline{V}_i \subset U_i$. Put $E_i = h^{-1}(\overline{U}_i)$ and $F_i = h^{-1}(\overline{V}_i)$. Further, let G_i: $p^{-1}(\overline{U}_i) \times I \to p^{-1}(\overline{U}_i)$ be a fiber homotopy from the identity to $\psi_i\phi_i$ and η_i: $B \to I$ be a map such that $\eta_i(B - U_i) = \{0\}$ and $\eta_i(\overline{V}_i) = \{1\}$.

Suppose that we have defined g_{i-1}: $X \to E$ such that g_{i-1} is homotopic to f over p and $\overline{g_{i-1}(\bigcup_{j<i} F_j)}$ is compact. Let $A = E_i \cap \bigcup_{j<i} F_j$ and let H_i: $E_i \times I \to \overline{U}_i \times F$ be a fiber homotopy such that $H_i(x, 0) = \phi_i g_{i-1}$ and $\overline{H_i((E_i \times \{1\}) \cup (A \times I))}$ is compact. Such H_i exist since F has RCP($\mathfrak{T}$) and $\overline{U}_i$ is compact.

Define g_i: $X \to E$ by

$$g_i(x) = \begin{cases} G_i(g_{i-1}(x), 2\eta_i h(x)), & \eta_i h(x) \in \left[0, \frac{1}{2}\right], \\ \psi_i H_i(x, 2\eta_i h(x) - 1), & \eta_i h(x) \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then g_i is homotopic to g_{i-1} (and hence to f) over p and $\overline{g_i(\bigcup_{j<i} F_j)}$ is compact as it is contained in $\overline{g_{i-1}(\bigcup_{j<i} F_j)} \cup \overline{\psi_i H_i((X \times \{1\}) \cup (A \times I))}$. Putting $g_0 = f$, the result follows by induction up to n.

PROOF OF LEMMA 2. Let Y be a CW-complex of finite type and let $\phi : Y \rightleftarrows K : \psi$ be a homotopy equivalence and its inverse, where K is a locally finite simplicial complex.

Suppose that X is a finite-dimensional normal space, A a subspace of X and f: $X \to Y$ a map such that $\overline{f(A)}$ is compact. Let $\mathcal{V}$ be the star cover of K and $\mathcal{U}$ a finite-dimensional cover of X that refines $(\phi f)^{-1}\mathcal{V}$. Let π: $X \to \nu\mathcal{U}$ be a canonical projection of X onto the nerve of $\mathcal{U}$. Then there exists a simplicial map σ: $\nu\mathcal{U} \to K$ such that $\sigma\pi$ is contiguous to ϕf.

Let Θ: $X \times I \to K$ be the linear deformation (see [3, p. 354]), ϕf to $\sigma\pi$ then $\Theta(A \times I) \cup \sigma\pi(X)$ is contained in some m-skeleton K^m of K. Let D: $Y \times I \to Y$ be a homotopy from the identity to $\psi\phi$. Define H: $X \times I \to Y$ by

$$H(x, t) = \begin{cases} D(f(x), 2t), & t \in \left[0, \frac{1}{2}\right], \\ \psi\Theta(x, 2t - 1), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We may assume that ψ is cellular so that $\psi(K^m) \subset Y^m$, the m-skeleton of Y, which is compact. Hence $H((X \times \{1\}) \cup (A \times I))$ is contained in a compact subset of Y, namely $Y^m \cup D(\overline{f(A)} \times I)$.

REMARKS. 1. A slight modification of the proof of Lemma 2 shows that for any given X one only needs that F has a compact (dim X)-skeleton.

2. The nature of the proof of Lemma 2 seems to indicate that Theorem 1 contains all the useful geometric information about the relation between homotopy and uniform homotopy in that it shows that it is very unlikely that there are other useful categories $\mathfrak{S}$ and $\mathfrak{T}$ such that all the spaces in $\mathfrak{S}$ have RCP($\mathfrak{T}$).

3. Part 2 of Theorem 1 is slightly stronger than (3.3) of **[1]** in that we do not require that B has the homotopy type of a CW-complex.

As usual β will denote the Stone-Čech compactification functor on the category of completely regular Hausdorff spaces.

THEOREM 2. *If Y is compact and $\pi_1 Y$ is infinite then there is a homotopically nontrivial map from $\beta\mathbf{R}$ to Y. Hence $[\beta -, Y]$ is not a homotopy functor on any category that contains the real line* $\mathbf{R}$.

PROOF. Let PY denote the space of paths in Y starting at $* \in Y$ and $p: PY \to Y$ the map $p(\lambda) = \lambda(1)$. Then p is a fibration with fiber ΩY, the space of loops at $*$. That a map $\beta f: \beta\mathbf{R} \to Y$ is homotopically trivial is equivalent to being able to factor it though p. This in turn is equivalent to being able to factor $f: \mathbf{R} \to Y$ through p via a bounded map into PY, **[1]**.

Since $\pi_1 Y$ is infinite, ΩY has infinitely many path components. Let $\{\sigma_i\}_{i=0}^{\infty} \subset \Omega Y$ be such that σ_0 is the constant loop to $*$ and σ_i and σ_j are in distinct path components for $i \neq j$. Define $f: \mathbf{R} \to Y$ by $f(x) = \sigma_i \sigma_{i-1}^{-1}(x - i)$, $x \in [i, i + 1]$ and $f(x) = *$, $x \leqslant 1$. Since Y is compact f extends to $\beta\mathbf{R}$.

Now any lift ϕ of f to PY must be unbounded as $\phi(i)$ and $\phi(j)$ must be in distinct path components of ΩY.

REMARK 4. The condition that Y is compact is essential in Theorem 2, since by **[2**, Theorem 3.4] for torsion abelian groups G, $[\beta -, K(G, 1)]$ is a homotopy functor on completely regular Hausdorff spaces, where $K(G, 1)$ is an Eilenberg-Mac Lane space of type $(G, 1)$. In particular one could take $G = \mathbf{Q}/\mathbf{Z}$.

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