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BOUNDARY PRESERVING MAPS OF 3-MANIFOLDS

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ABSTRACT. We prove an extension of Waldhausen's theorem [5] conjectured by Hempel in [3].

We prove the following extension of Waldhausen's theorem [5]:

THEOREM 1. *Let M, N be P^2-irreducible 3-manifolds. Suppose that M is compact, sufficiently large and f: $(M, \partial M) \to (N, \partial N)$ is a continuous map inducing an injection f_*: $\pi_1(M) \to \pi_1(N)$. Then, there is a proper homotopy f_t: $(M, \partial M) \to (N, \partial N)$ such that $f_0 = f$ and either*

(i) *f_1: $M \to N$ is a covering map, or*

(ii) *M is an I-bundle over a closed surface, and $f_1(M) \subset \partial N$, or*

(iii) *N (hence also M) is a solid torus or a solid Klein bottle and f_1: $M \to N$ is a branched covering with branch set a circle, or*

(iv) *M is a cube with handles and $f_1(M) \subset \partial N$.*

If $f|B$: $B \to C$ is already a covering map, we may assume $f_t|B = f|B$, for all t (where B is any component of ∂M and C the component of ∂N containing $f(B)$).

This theorem is conjectured in [3], where it is proved under additional restrictions. When M, N are orientable and f_* is an isomorphism, a variant of this result is proved by Evans in [2]. Our argument yields a simple proof of his result too. We refer to [1] and [4] for the concepts 'geometric degree', 'absolute degree', 'orientation-true', etc. The term 'degree of a map f' will be used for the twisted degree of f and will be denoted by $\deg f$ as in [4].

LEMMA 2. *Let M be a compact irreducible 3-manifold and let f: $(M, \partial M) \to (N, \partial N)$ be a map into any aspherical 3-manifold such that f_*: $\pi_1(M) \to \pi_1(N)$ is injective. Let S be a component of ∂M, S' the component ∂N containing $f(S)$. If the geometric degree of $(f|S)$: $S \to S'$ is zero, then M is a cube with handles and f is properly homotopic to a map into ∂N.*

REMARK. If $(f|S)_*(\pi_1(S))$ is free subgroup of $\pi_1(S')$, then the geometric degree of $f|S$ is zero.

PROOF. Since the geometric degree of $(f|S)$ is zero, $(f|S)$ is homotopic to a map of S into S'-P for any $p \in S'$. Hence $(f|S)_*(\pi_1(S)) = H$ is a finitely generated free subgroup of $\pi_1(S'-p)$. Representing H by the fundamental group of a wedge X of circles, we can write $(f|S)$, up to homotopy, as a composite $S \xrightarrow{g} X \xrightarrow{h} S' - p$.

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Making g transverse to a point of X ($\neq$ base point of X) we can find a nontrivial simple loop l in S such that $g(l)$ is null-homotopic in X and hence $f(l)$ is null-homotopic in $S' - p$. Since $f_*: \pi_1(M) \to \pi_1(N)$ injective, l bounds a disc D in M. Split M along D to obtain M_1, M_2 (one of which may be empty) and maps g_i: $M_i \to N$. Since $f(l)$ is null-homotopic in $S' - p$ and since N is aspherical we may assume that g_i maps $(M_i, \partial M_i)$ to $(N, \partial N)$, $i = 1, 2$. If S_i is the component of ∂M_i obtained from S, g_i maps S_i into $S' - p$ and therefore the degree of $(g_i|S_i)$ is zero. Induction completes the proof of Lemma 2.

The following two lemmas follow easily from [**1**] and [**4**]; we sketch the proofs:

LEMMA 3. *Let*

$$\begin{array}{ccc} (\tilde{M}, \partial\tilde{M}) & \xrightarrow{\tilde{f}} & (\tilde{N}, \partial\tilde{N}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ (M, \partial M) & \xrightarrow[f]{} & (N, \partial N) \end{array}$$

be a commutative diagram, where p, q are covering projections with the same finite number of sheets and $f_*: \pi_1(M) \to \pi_1(N)$ is surjective. Then the geometric degree of f is equal to the geometric degree of $\tilde{f}$.

PROOF. Let $G(f)$, $A(f)$ and $a(f, 2)$ denote the geometric degree, absolute degree, and the mod 2 degree of f and similarly for $\tilde{f}$. By a theorem of Hopf (see [**1**]), $G(f) = A(f)$. By Theorem 3.1 of [**1**], $A(f) = a(f, 2)$ mod 2. From the definition of geometric degree it is immediate that $G(f) \geq G(\tilde{f})$ and $G(f) = G(\tilde{f})$ mod 2. This together with the above assertions implies that $G(f) = G(\tilde{f})$ if $G(f) \leq 1$. Hence, it remains to consider the case when $G(f) > 1$. From the classification into three types used in defining absolute degree (see [**1**, p. 371] and [**4**, p. 375]), if f is of type II or III, then $G(f) = A(f) \leq 1$. Since $G(f) > 1$, f has to be of type I and hence it is orientation-true. In this case, by Theorem VIII of [**4**], $A(f) = |\deg f|$. By Lemma 3.6 of [**4**],

$$|\deg f|\ |\deg p| = |\deg(fp)| = |\deg(q\tilde{f})| = |\deg q|\ |\deg \tilde{f}|.$$

Since, by assumption $|\deg p| = |\deg q|$, we have $|\deg f| = |\deg \tilde{f}|$ and therefore $G(f) = G(\tilde{f})$ in the last case too.

LEMMA 4. *Let f: $(M, \partial M) \to (N, \partial N)$ be a map inducing isomorphisms in the fundamental groups. If the geometric degree of f is nonzero, then $f_*: \pi_1(M) \to \pi_1(N)$ is orientation-true* (*and the modulus of the degree of f is equal to the geometric degree of f*).

PROOF. Since $G(f) \neq 0$ and since f has no kernel, f is of type I and hence is orientation-true. Hence $G(f) = |\deg f|$ by Theorem VIII of [**4**].

Lemma 2 is what is needed to extend the argument of [**3**] to the general case; we will quickly sketch the proof along the lines of proof of Theorem 13.6 of [**3**].

Proof of Theorem 1. Since we are not assuming that N is compact, the theorem will follow from the case when f_* is an isomorphism. From now on we will assume that f_* is an isomorphism. We will show that either (ii) or (iii) or (iv) of Theorem is valid or f is orientation-true and the degree of f is ± 1. Then by the extension of Lemma 3.1 of [2] to the nonorientable case, Theorem 1 follows.

By Lemma 2, either (iv) is valid or $(f|S): S \to S'$ has nonzero geometric degree for each component S of ∂M; here S' denotes the component of ∂N containing $f(S)$. In the latter case $(f|S)_*\pi_1(S)$ is a subgroup of finite index in $\pi_1(S')$. Thus, if two components S, T of ∂M map into S', then $(f|S)_*\pi_1(S)$ and $(f|T)_*\pi_1(T)$ intersect in a subgroup of finite index. Now one sees as in [3], that M is a product I-bundle and (ii) is valid. Thus, either (iv) or (ii) is valid or, f satisfies

(p.1) $(f|S)_*\pi_1(S)$ is of finite index in $\pi_1(S')$ where S is any component of ∂M and S' is the component of ∂N containing $f(S)$; and

(p.2) the map $\pi_0(\partial M) \to \pi_0(\partial N)$ induced by f is injective.

From now on we will assume that f satisfies (p.1) and (p.2). Construct a commutative diagram

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\bar{f}} & \bar{N} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{f} & N \end{array}$$

when $\bar{M}$, $\bar{N}$ are orientable and p, q are finite covers of the same degree $\leqslant 4$. Since f satisfies (p.1), $\bar{f}$ also satisfies (p.1). If $\bar{f}$ does not satisfy (p.2), it follows as above that $\bar{M}$ is a product I-bundle. Since $\bar{M}$, $\bar{N}$ are orientable, $\bar{N}$ is also a product I-bundle and $\bar{f}$ can be properly deformed into $\partial\bar{N}$. Since f satisfies (p.2), it follows that M is a nontrivial I-bundle and that $f_*(\pi_1(M))$ is peripheral in N. Hence f can be properly deformed into ∂N. Hence either (ii) holds or $\bar{f}$ satisfies (p.1) and (p.2). Thus, we see that either (iv) or (ii) holds or both f and $\bar{f}$ satisfy (p.1) and (p.2). In the later case we choose local orientations for M, N and orient $\bar{M}$, $\bar{N}$ accordingly. Since $\bar{f}$ satisfies (p.1), (p.2) and since $\bar{M}$, $\bar{N}$ are orientable, we see that the degree of $\bar{f}$ is nonzero. If $\chi(M) \neq 0$, then $\chi(\bar{M}) \neq 0$ and the degree of $\bar{f}$ has to be ± 1 (see the argument [3, pp. 146–147]). Now Lemmas 3 and 4 show that the degree of f is ± 1 and by the extension of Lemma 3.1 of [2], $f|\partial M$ can be deformed to a homeomorphism. Hence f can be deformed to a homeomorphism as in [3] and (i) holds. It remains to consider the case when $\chi(M) = \chi(\bar{M}) = 0$ and $\deg f \neq \pm 1$. We claim that $\bar{S}$ is compressible in $\bar{M}$. Otherwise, since the degree of $(\bar{f}|\bar{S}) \neq \pm 1$, $(\bar{f}|\bar{S})_*\pi_1(\bar{S})$ is a proper rank two subgroup of $\pi_1(\bar{f}(\bar{S}))$ and it follows that $\bar{M}$, $\bar{N}$ are nontrivial I-bundles. In this case $(\bar{f}|\bar{S})_*$ has to be an isomorphism. This contradiction shows that $\bar{S}$ is compressible in $\bar{M}$ and it follows as in [3] that (iii) holds. This completes the proof of Theorem 1.

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