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A DIRECT SUMMAND IN $H^*(MO\langle 8\rangle, Z_2)$

A. P. BAHRI AND M. E. MAHOWALD[1]

ABSTRACT. $H^*(MO\langle 8\rangle, Z_2)$ as a module over the Steenrod algebra is shown to have a direct summand $A//A_2 \cdot U$.

In this note we show that, as a module over the Steenrod algebra A, $H^*(MO\langle 8\rangle, Z_2)$ has a direct summand beginning in dimension 0. The proof is easy but contradicts the theorem of Giambalvo [3]. Recall that $MO\langle 8\rangle$ is the Thom space of the bundle induced from the canonical bundle over BO by p: $BO\langle 8\rangle \to BO$ the projection of the 7-connected covering. A cobordism theory $\Omega^{\langle 8\rangle}$ results from considering $MO\langle 8\rangle$ as a spectrum in the usual way. For some partial computations and further details the reader is referred to [2].

Let A_2 be the augmentation ideal of the Hopf subalgebra of A generated by $\{Sq^0, Sq^1, Sq^2, Sq^4\}$. Denote by $A//A_2$ the quotient coalgebra A/AA_2. Let U be the Thom class in $H^*(MO\langle 8\rangle)$. All homology groups are to have Z_2 coefficients.

THEOREM. *$A//A_2 \cdot U$ is a direct summand in $H^*MO\langle 8\rangle$.*

PROOF. The argument follows similar lines to Priddy's proof that $K(Z_2)$ is a Thom spectrum [5]. Let X denote the 15-skeleton of $BO\langle 8\rangle$ and i: $X \to BO\langle 8\rangle$ the inclusion. Since $BO\langle 8\rangle$ is a double loop space there is an induced double loop map

$$\omega: \Omega^2\Sigma^2 X \to \Omega^2\Sigma^2 BO\langle 8\rangle \to BO\langle 8\rangle$$

where the first map is $\Omega^2\Sigma^2 i$ and the second is the adjoint of the identity double looped. Let α: $A//A_2 \to H^*MO\langle 8\rangle$ denote evaluation on the Thom class and Φ_*: $H_*BO\langle 8\rangle \to H_*MO\langle 8\rangle$ the Thom isomorphism α_*, the dual of α, is a morphism of algebras over A_*, the dual of the Steenrod algebra. Now $\Phi_*\omega_*H_*\Omega^2\Sigma^2X$ is a subalgebra over A_* of $H_*MO\langle 8\rangle$ since it is equal to $\Gamma_*H_*M(p\omega)$ where $M(p\omega)$ is the Thom spectrum associated with $p\omega$: $\Omega^2\Sigma^2X \to BO$ and Γ: $M(p\omega) \to MO\langle 8\rangle$ is the map induced by ω.

To prove the theorem it will be enough to show that

$$\alpha_*: \Phi_*\omega_*H_*\Omega^2\Sigma^2X \to (A//A_2)_*$$

is an algebra isomorphism where $(A//A_2)_*$ is the dual of $A//A_2$. To do this we need to know about the image of ω_*.

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Lemma. *$\omega_* H_* \Omega^2\Sigma^2 X \approx Z_2[p_8, p_{12}, p_{14}, Q_1^n(p_{15}), n \geqslant 0]$ where p_i is the nonzero primitive element in $H_i BO\langle 8\rangle$ and Q_1^n is the nth iterate of the Dyer-Lashof operation defined on a double loop space. Q_1^0 = the identity and* dim $Q_1^n(p_{15}) = 2^{n+4} - 1$.

Proof. The structure of $H^* BO\langle 8\rangle$ has been computed by Stong [6, Theorem A] and is given by

$$H^* BO\langle 8\rangle \approx H^* K(Z, 8)/ASq^2 \otimes Z_2[\theta_i]$$

where the θ_i are classes in $H^* BO \equiv w_i$ mod decomposables. The first is θ_{16} in dimension 16. It follows then that X is a four-cell complex with cells in dimensions 8, 12, 14 and 15 corresponding to the classes

$$x_8 = \varepsilon_8, \qquad x_{12} = Sq^4\varepsilon_8, \qquad x_{14} = Sq^6\varepsilon_8 = Sq^2Sq^4\varepsilon_8,$$
$$x_{15} = Sq^7\varepsilon_8 = Sq^1Sq^2Sq^4\varepsilon_8,$$

where ε_8 is the first class in $H^* K(Z, 8)$. We will denote by p_i the class in homology dual to x_i.

We may now use Theorem 3 of Browder [1] to conclude that $H_* \Omega^2\Sigma^2 X$ is a polynomial ring over Z_2 generated by four types of elements. These are

$$p_i, \quad i = 8, 12, 14, 15; \qquad Q_1^n(p_i), \quad n > 0; \qquad \psi_1(p_i, y_j); \qquad Q_1^n(\psi_1(p_i, y_j))$$

where here we have identified p_i with its image under the inclusion $H_* X \subset H_* \Omega^2\Sigma^2 X$. ψ_1 is the Browder operation defined on a double loop space in [1] and the y_j's are iterated products involving ψ_1 in $H_* \Omega^2\Sigma^2 X$. They are determined by giving a basis for the graded Lie algebra generated by $H_* \Sigma X$ in its tensor algebra. Full details may be found in §IV of [1].

We will now analyse in turn what happens to each of the above elements under ω_*. The map

$$X \xrightarrow{\gamma} \Omega^2\Sigma^2 X \xrightarrow{\omega} BO\langle 8\rangle,$$

where γ is the adjoint of the identity on $\Sigma^2 X$, is just the inclusion of the 15-skeleton and we may safely identify p_i with its image under ω_*. These give primitive elements in $H_* BO\langle 8\rangle$.

Since ω is a double loop map, we have, by the naturality of ψ_1 in the category of double loop spaces, that

$$\omega_* \psi_1(p_i, y_j) = \psi_1\big(p_i, \omega_*(y_j)\big) = 0.$$

since the operation on the right is in $BO\langle 8\rangle$ which is a triple loop space and must have ψ_1 identically zero in it. This 'instability' of the Browder operations follows immediately from their definition in [1]. The $\omega_* Q_1^n \psi_1(p_i, y_j)$ are also all zero since the Q_1^n are natural with respect to ω.

One of the consequences of Stong's Theorem A is that the map $p^*: H^* BO \to H^* BO\langle 8\rangle$ is onto and hence that p_* is a monomorphism. We will now use this to determine the $\omega_* Q_1^n(p_i)$. Since $p: BO\langle 8\rangle \to BO$ is a double loop map we have

$$p_* \omega_* Q_1^n(p_i) = Q_1^n\big(p_*(p_i)\big) \qquad (n > 0).$$

The RHS is zero for $i = 8, 12, 14$ and nonzero for $i = 15$ by Kochman [4, Corollary 35], so $\omega_* Q_1^n(p_i) = 0$ for $i = 8, 12, 14$ and $\neq 0$ for $i = 15$.

We have shown therefore that the only generators of $H_*\Omega^2\Sigma^2 X$ which survive under ω_* are $p_8, p_{12}, p_{14}, p_{15}$ and $Q_1^n(p_{15})$ for $n > 0$. To finish the lemma we need to show that these elements generate a polynomial ring. We will do this and complete the proof of the theorem at the same time.

Now $(A//A_2)_* \approx Z_2[\xi_1^8, \xi_2^4, \xi_3^2, \xi_4, \xi_5, \ldots]$ where $\dim \xi_i = 2^i - 1$ and $Z_2[\xi_1, \xi_2, \xi_3, \xi_4, \ldots]$ is the dual of the Steenrod algebra. We know that, in each dimension, the rank of $\omega_* H_*\Omega^2\Sigma^2 X$ is not greater than that of $(A//A_2)_*$ and so the theorem follows from the next lemma.

LEMMA.

$$\omega_* H_*\Omega^2\Sigma^2 X \underset{\Phi_*}{\rightarrow} H_* MO\langle 8\rangle \underset{\alpha_*}{\rightarrow} (A//A_2)_*$$

is onto.

PROOF. It suffices to show that each of p_8, p_{12}, p_{14} and $Q_1^n(p_{15})$ is mapped to the generator in the corresponding dimension of $(A//A_2)_*$ modulo decomposable elements.

Now the duals of ξ_1^8, ξ_2^4 and ξ_3^2 in $A//A_2$ are Sq^8, Sq^{12} and Sq^{14} respectively, these being the only elements in their dimensions. Further,

$$\left\langle Sq^8, \alpha_*\Phi_*(p_8)\right\rangle = \left\langle Sq^8 U, \Phi_*(p_8)\right\rangle = \left\langle w_8 U, \Phi_*(p_8)\right\rangle$$
$$= \langle w_8, p_8\rangle = 1$$

and so $\alpha_*\Phi_*(p_8) = \xi_1^8$. Similarly $\alpha_*\Phi_*(p_{12}) = \xi_2^4$ and $\alpha_*\Phi_*(p_{14}) = \xi_3^2$.

Consider now the following commutative diagram:

$$\begin{array}{ccccc} H_*BO\langle 8\rangle & \overset{\Phi_*}{\rightarrow} & H_*MO\langle 8\rangle & \underset{\alpha_*}{\rightarrow} & (A//A_2)_* \\ p_*\downarrow & & P_*\downarrow & & \downarrow \\ H_*BO & \underset{\Phi_*}{\rightarrow} & H_*MO & \overset{\alpha_*}{\rightarrow} & A_* \end{array}$$

where P: $MO\langle 8\rangle \rightarrow MO$ is the map of Thom spectra induced from p. For each n, $p_*(Q_1^n(p_{15})) = Q_1^n(p_*(p_{15})) = Q_1^n(p'_{15})$ where p'_{15} is the nontrivial primitive element in $H_{15}BO$. By Kochman's result [4, Corollary 35], we have $Q_1^n(p'_{15}) = p'_{2^{n+4}-1}$ the nonzero primitive in $H_{2^{n+4}-1}BO$. Now it is shown in the last paragraph of [5] that

$$\alpha_*\Phi_*(p'_{2^{n+4}-1}) \equiv \xi_{n+4} \quad \text{mod decomposables}$$

and so, by the commutativity of the diagram,

$$\alpha_*\Phi_* Q_1^n(p_{15}) \equiv \xi_{n+4} \quad \text{mod decomposables.}$$

This completes the proof of the lemma and of the theorem.

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MATHEMATICAL INSTITUTE, 24–29 ST. GILES, OXFORD OX1 3LB, ENGLAND

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201

ABCDEFGHIJ–CM–89876543210