MODULES WITH ARTINIAN PRIME FACTORS

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Abstract. An $R$-module $M$ has Artinian prime factors if $M/PM$ is an Artinian module for each prime ideal $P$ of $R$. For commutative rings $R$ it is shown that Noetherian modules with Artinian prime factors are Artinian. If $R$ is either commutative or a von Neumann regular $V$-ring then the endomorphism ring of a module with Artinian prime factors is a strongly $\pi$-regular ring.

A ring $R$ with 1 is left $\pi$-regular if for each $a \in R$ there is an integer $n > 1$ and $b \in R$ such that $a^n = a^{n+1}b$. Right $\pi$-regular is defined in the obvious way, however a recent result of F. Dischinger [5] asserts the equivalence of the two concepts. A ring $R$ is $\pi$-regular if for any $a \in R$ there is an integer $n > 1$ and $b \in R$ such that $a^n = a^nba^n$. Any left $\pi$-regular ring is $\pi$-regular but not conversely. Because of this, we say that $R$ is strongly $\pi$-regular if it is left (or right) $\pi$-regular.

In [2, Theorem 2.5] it was established that if $R$ is a (von Neumann) regular ring whose primitive factor rings are Artinian and if $M$ is a finitely generated $R$-module then the endomorphism ring $\text{End}_R(M)$ of $M$ is a strongly $\pi$-regular ring. Curiously enough, the same is not true for finitely generated modules over strongly $\pi$-regular rings, as Example 3.1 of [2] shows. Obviously, it also fails for arbitrary regular rings. These observations lead one to consider conditions on finitely generated modules which ensure that the endomorphism ring is strongly $\pi$-regular. A natural one seems to be that of having Artinian prime factors. In fact, we establish that such modules have strongly $\pi$-regular endomorphism ring whenever the base ring is either commutative or a regular $V$-ring.

Consider a finitely generated module $M$ over a ring $R$. If $R$ is commutative then $R$ is $\pi$-regular if and only if its prime ideals are maximal [11]. Accordingly, when $R$ is commutative and $\pi$-regular, $M/PM$ is an Artinian module for all primes $P$. This observation serves as a starting point.

Theorem 1. Suppose $R$ is a commutative ring. For a finitely generated $R$-module $M$ the following conditions are equivalent.

(a) $M$ has Artinian prime factors.
(b) $S = \text{End}_R(M)$ is a strongly $\pi$-regular ring.
(c) $R/\text{Ann}_R(M)$ is a $\pi$-regular ring.

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PROOF. (b) \implies (c): Since $M$ is finitely generated, $S$ satisfies a polynomial identity. Hence each prime ideal of $S$ is a maximal ideal (see e.g. [1]). Now $R' = R/\text{Ann}_R(M)$ embeds in the center of $S$ and $S$ is an integral extension of $R'$ [10]. It follows that prime ideals of $R'$ are maximal ideals [3] and hence $R'$ is $\pi$-regular.

(c) \implies (a): This is clear.

(a) \implies (b): By [2, Proposition 2.3], $S$ is strongly $\pi$-regular if and only if for each $\alpha \in S$ there is an integer $t > 1$ such that $M = \text{Ker}\alpha' \oplus M\alpha'$. Thus let $\alpha \in S$. Our first step is to show that there is an integer $t > 1$ such that $M\alpha' = M\alpha'^{t+1}$. Assume that no such integer exists. Because $M$ is finitely generated, there is an ideal $P$ of $R$ which is maximal among those ideals $I$ of $R$ having the property that $M^k \subseteq M^{k+1} + IM$ for all integers $k \geq 1$. We claim that $P$ is a prime ideal as claimed. By assumption, $M/PM$ is an Artinian module. But then the sequence of submodules $M^1 \supseteq M^2 \supseteq \cdots$ must terminate modulo $PM$, providing the desired contradiction. This shows then that $M\alpha' = M\alpha'^{t+1}$ for some integer $t > 1$. Now $\alpha$ is an onto endomorphism of the finitely generated $R$-module $M\alpha'$, and so $\alpha$ is 1-1 on $M\alpha'$ since $R$ is commutative [12]. Then $\text{Ker}\alpha' = 0$ implies that $\text{Ker}\alpha' = \text{Ker}\alpha'^t$. It now follows easily that $M = \text{Ker}\alpha' \oplus M\alpha'$, completing the proof.

Examination of the proof of the implication (a) \implies (b) shows that commutativity was used only to ensure that onto endomorphisms are 1-1. It has been shown in [2, Theorem 2.2] that rings integral over their center and satisfying a polynomial identity have the property that onto endomorphisms of finitely generated modules are 1-1, a property which left Noetherian rings also have. Thus we are able to state the following.

**Theorem 2.** Assume $R$ is either a PI-ring integral over its center or a left Noetherian ring. If $M$ is a finitely generated left $R$-module having Artinian prime factors then $\text{End}_R(M)$ is a strongly $\pi$-regular ring.

In view of this theorem one might ask if any Noetherian $R$-module with Artinian prime factors is Artinian. The answer is no, in general. An example in [9, p. 66] provides us with a perfect ring $D$ having a Noetherian non-Artinian module. Since $D/P$ is simple Artinian for any prime ideal $P$, such a module must have Artinian prime factors. Before showing that the answer is affirmative when $R$ is commutative, we note that over a semiprimary ring, all Noetherian modules are Artinian, so the answer is (trivially) yes in this case.

**Theorem 3.** If $R$ is a commutative ring and $M$ is a Noetherian $R$-module with Artinian prime factors then $M$ is Artinian.
Proof. By Theorem 1, \( R' = R/\text{Ann}_R(M) \) is a \( \pi \)-regular ring. Since \( M \) is a faithful finitely generated \( R' \)-module, \( R' \) is isomorphic to a submodule of a finite direct sum of copies of \( M \). Hence \( R' \) is a Noetherian module. Any Noetherian \( \pi \)-regular ring is Artinian so \( R' \) is Artinian. But then \( M \), being a finitely generated \( R' \)-module, must be Artinian.

While it is false in general that Noetherian modules with Artinian prime factors are Artinian, the following is true.

**Theorem 4.** If \( M \) is a Noetherian \( R \)-module with Artinian prime factors then \( S = \text{End}_R(M) \) is semiprimary.

**Proof.** The proof of (a) \( \Rightarrow \) (b) shows that \( S \) is a strongly \( \pi \)-regular ring. Thus each non-nil one sided ideal contains a nonzero idempotent. It follows that \( J(S) \), the Jacobson radical of \( S \), is a nil ideal. By a theorem of L. Small (see [6, Theorem 2.1]), nil subrings of \( S \) are nilpotent, so that \( J(S) \) is a nilpotent ideal of \( S \). Now orthogonal idempotents of \( S/J(S) \) can be lifted to \( S \). However \( M \) is Noetherian so \( S \) can have no infinite set of orthogonal idempotents. It follows then that \( S/J(S) \) is a semisimple Artinian ring.

This theorem generalizes the well-known fact that the endomorphism ring of a Noetherian Artinian module is semiprimary.

We now turn to a result which covers [2, Theorem 2.3]. Recall that a (left) \( V \)-ring is a ring all of whose simple left modules are injective. For the salient features of \( V \)-rings we refer the reader to [4, Chapter 5].

**Theorem 5.** Assume \( R \) is a regular \( V \)-ring. If \( M \) is a finitely generated \( R \)-module with Artinian prime factors then \( S = \text{End}_R(M) \) is a strongly \( \pi \)-regular ring.

**Proof.** Let \( \alpha \in S \). As in the proof of Theorem 1, there is an integer \( t > 1 \) such that \( Ma^t = Ma^{t+1} \). Suppose \( x \in \ker \alpha^{t+1} \); if \( u = xa^t \neq 0 \), then there is an ideal \( P \) of \( R \) maximal among those ideals \( I \) of \( R \) for which \( u \in IM \). Hence \( u \in AM \) for all ideals \( A \) of \( R \) properly containing \( P \). If \( P \) is not a prime ideal then there are ideals \( A \) and \( B \) of \( R \) properly containing \( P \) for which \( AB \subseteq P \). Then \( u \in BM \) so that \( Au \subseteq ABM \subseteq PM \). Since we also have \( u \in AM \) we can write \( u = \Sigma a_i m_i \) where \( a_i \in A \), \( m_i \in M \). Because \( R \) is a regular ring there is an idempotent \( e \in A \) such that \( ea_i = a_i \) for each \( i \). But then \( u = eu \in Au \subseteq PM \), a contradiction. Thus \( P \) must be a prime ideal and the module \( M/PM \) is Artinian. Because \( R \) is a \( V \)-ring and \( M/PM \) has finitely generated essential socle, we infer that \( M/PM \) is completely reducible and hence Noetherian. Then \( \alpha \) induces \( \beta \in \text{End}_R(M/PM) \) and \( (M/PM)\beta^t = (M/PM)\beta^{t+1} \) and this yields \( \ker \beta^t = \ker \beta^{t+1} \). But then \( xa^t \in PM \), which is the desired contradiction. It now follows that \( \ker \alpha^t = \ker \alpha^{t+1} \), \( M = Ma^t \oplus \ker \alpha^t \), and so \( S \) is strongly \( \pi \)-regular.

**Corollary 5** [2, Theorem 2.3]. If \( R \) is a regular ring whose primitive factor rings are Artinian then \( \text{End}_R(M) \) is strongly \( \pi \)-regular for any finitely generated \( R \)-module \( M \).
Proof. It is enough to note that (i) $R$ is a $V$-ring, and (ii) prime factor rings of $R$ are Artinian. That (i) holds follows from [4, Corollary 5.13] while [8, Theorem 3, p. 239] guarantees (ii).

It is straightforward to see that a finitely generated projective Artinian module over a semiprime ring is completely reducible. Thus the proof of Theorem 4 can be used to prove the next result.

**Theorem 6.** Let $R$ be a regular ring and $M$ a finitely generated $R$-module. If $M/PM$ is a projective Artinian $R/P$-module for each prime ideal $P$ of $R$ then $\text{End}_R(M)$ is strongly $\pi$-regular.

In the first version of this article we asked whether or not a finitely generated Artinian module over a regular ring is Noetherian. An affirmative answer would then imply the statement,

over any regular ring, finitely generated modules with Artinian prime factors have a strongly $\pi$-regular endomorphism ring.

Recently, K. Goodearl has constructed examples of cyclic Artinian non-Noetherian modules as well as Noetherian non-Artinian modules over regular rings [7]. Thus our original question has a negative answer. However the validity of (*) still remains open, and would be true should the following question have a positive response. If $M$ is a finitely generated Artinian module over a regular ring, is every onto endomorphism of $M$ also 1-1?

**References**


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