PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 3, March 1980

THE RADICAL OF THE CENTER OF A GROUP ALGEBRA

D. S. PASSMAN

ABSTRACT. Let $K[G]$ denote the group algebra of a finite group G over a field K of characteristic $p > 0$ and let $\mathfrak{Z} = \mathbf{Z}(K[G])$. In this paper, we offer a bound for the nilpotence degree of the Jacobson radical $J\mathfrak{Z}$ in terms of the order of a Sylow p-subgroup of G.

Let $K[G]$ denote the group algebra of the finite group G over the field K. If $|G| \neq 0$ in K, then by Maschke's theorem [**1**, Theorem 2.4.2], $K[G]$ and $\mathfrak{Z} = \mathbf{Z}(K[G])$ are both semisimple. On the other hand, if $|G| = 0$ in K, then $\hat{G} = \sum_{x \in G} x$ is a nonzero central element of $K[G]$ of square zero and hence both $K[G]$ and $\mathfrak{Z}$ are not semisimple. In this paper, we study the Jacobson radical of $\mathfrak{Z}$ and we bound its nilpotence degree. In view of the above remarks, we can assume that char $K = p > 0$. The main result is then:

THEOREM. *Let* char $K = p > 0$ *and let* $|G| = p^a b$ *with* $p \nmid b$. *If* $J\mathfrak{Z}$ *denotes the Jacobson radical of* $\mathfrak{Z} = \mathbf{Z}(K[G])$, *then*

$$(J\mathfrak{Z})^{(p^{a+1}-1)/(p-1)} = 0.$$

The proof of this is based on ideas of W. Willems and in particular on [**2**, Hilfsatz 4.3]. However here we work totally within $K[G]$ rather than dealing with central characters in characteristic zero.

We start by listing notation and a few basic facts (see [**1**, pp. 138–141]). Let D be a p-subgroup of G.

(1) $\mathfrak{I}(D)$ is the span of all class sums of G with defect group $\subseteq_G D$. By [**1**, Lemma 4.3.12], $\mathfrak{I}(D)$ is an ideal of $\mathfrak{Z}$.

(2) $\tilde{\mathfrak{I}}(D)$ is the span of all class sums of G with defect group $<_G D$. It follows from the above that $\tilde{\mathfrak{I}}(D)$ is also an ideal of $\mathfrak{Z}$ which is clearly contained in $\mathfrak{I}(D)$.

(3) $\pi_{\mathbf{C}(D)}$ is the natural projection map $\pi_{\mathbf{C}(D)}$: $K[G] \to K[\mathbf{C}(D)]$. By [**1**, Theorem 4.3.10], $\pi_{\mathbf{C}(D)}$ induces a homomorphism, the Brauer homomorphism, from $\mathbf{Z}(K[G])$ to $\mathbf{Z}(K[\mathbf{C}(D)])$.

(4) If $g \in G$, we write $g = g_{p'} g_p$ as the product of its p'- and p-parts. We use $\sim$ to denote conjugates in G and we use $d(\cdots)$ to denote defect groups. Furthermore $\omega(K[D])$ is the augmentation ideal of $K[D]$.

Received by the editors December 8, 1978.
AMS (MOS) subject classifications (1970). Primary 20C05.

LEMMA 1. *Let D be a central p-subgroup of G. If $g, h \in G$ we write $g \approx h$ if and only if $gD \sim hD$, that is, if and only if these elements are conjugate in G/D.*

(i) *$\approx$ is an equivalence relation and if $g \approx h$ then $d(g) \sim d(h)$.*

(ii) *Let $g \in G$ with $d(g) = D$ and let $p^n > |G|$. Then $h^{p^n} \sim g^{p^n}$ if and only if $h \approx g$.*

(iii) *Let $\beta \in K[G]$ be nilpotent with $\beta = \Sigma b_x x$. If $d(g) = D$, then $\Sigma_{h \approx g} b_h = 0$.*

PROOF. (i) It is clear that $\approx$ is an equivalence relation. Furthermore, since D is central, $g \approx h$ implies that $\mathbf{C}(g) \sim \mathbf{C}(h)$ and hence $d(g) \sim d(h)$.

(ii) Let $d(g) = D$. Then g_p is a p-element commuting with g so $g_p \in D \subseteq \mathbf{Z}(G)$. Thus $\mathbf{C}(g_{p'}) = \mathbf{C}(g)$ and $d(g_{p'}) = D$. Now suppose $h^{p^n} \sim g^{p^n}$. Then since $p^n > |G|$ we have

$$(h_{p'})^{p^n} = h^{p^n} \sim g^{p^n} = (g_{p'})^{p^n}$$

so $h_{p'} \sim g_{p'}$ and hence $d(h_{p'}) \sim d(g_{p'}) = D$. But h_p and g_p are p-elements commuting with $h_{p'}$ and $g_{p'}$ respectively so $h_p, g_p \in D \lhd G$ and $h \approx h_{p'} \approx g_{p'} \approx g$. Conversely if $h \approx g$, then $h \sim gd$ for some $d \in D \subseteq \mathbf{Z}(G)$ and $h^{p^n} \sim g^{p^n} d^{p^n} = g^{p^n}$.

(iii) Let $\beta \in K[G]$ be nilpotent and let $d(g) = D$. Choose p^n so that $p^n > |G|$ and $\beta^{p^n} = 0$. If $\beta = \Sigma b_x x$, then by [**1**, Lemma 2.3.1]

$$0 = \beta^{p^n} = \sum b_x^{p^n} x^{p^n} + \gamma$$

for some $\gamma \in [K[G], K[G]]$. By [**1**, Lemma 2.3.2] the sum of the coefficients in γ of terms conjugate to g^{p^n} is zero. Hence we have

$$0 = \sum_{h^{p^n} \sim g^{p^n}} b_h^{p^n} = \left(\sum_{h^{p^n} \sim g^{p^n}} b_h \right)^{p^n}.$$

But by (ii) above, $h^{p^n} \sim g^{p^n}$ if and only if $h \approx g$, so we obtain $\Sigma_{h \approx g} b_h = 0$.

LEMMA 2. *Let D be a central p-subgroup with $|D| = p^d$. Then*

(i) *$\mathcal{I}(D) \cdot J\mathfrak{Z} = \omega(K[D]) \cdot I(D)$,*

(ii) *$\mathcal{I}(D) \cdot (J\mathfrak{Z})^{p^d} = 0$.*

PROOF. (i) Observe that $\omega(K[D]) \subseteq \mathfrak{Z}$ and that $\omega(K[D])$ is nilpotent, by [**1**, Lemma 3.1.6]. Hence $\omega(K[D]) \subseteq J\mathfrak{Z}$ and we have $\mathcal{I}(D) \cdot J\mathfrak{Z} \supseteq \omega(K[D]) \cdot \mathcal{I}(D)$.

We now consider the reverse inclusion and we use the notation of the preceding lemma. Let $y \in G$ with $d(y) = D$ and let $\alpha \in J\mathfrak{Z}$. Then α is nilpotent and central so $\beta = y\alpha$ is nilpotent. Write

$$\beta = \beta_1 + \beta_2 + \cdots + \beta_s + \gamma$$

where each β_j is the sum of those terms in β belonging to a single equivalence class under $\approx$ and with elements having defect group D. Furthermore γ is the sum of the remaining terms where all defect groups are strictly larger than D, since D is central.

Fix coset representatives $z_1, z_2, \ldots, z_k$ for D in G and for any $\delta \in K[G]$ define $\delta^* = \Sigma_i \delta^{z_i}$. Then, since D is central, δ^* is independent of the choice of representatives z_i and hence $\delta^* \in \mathfrak{Z}$. Clearly

$$\beta^* = \beta_1^* + \beta_2^* + \cdots + \beta_s^* + \gamma^*.$$

Note that α is central and $\beta = y\alpha$ so

$$\beta^* = y^*\alpha = [\mathbf{C}(y) : D]\hat{\mathfrak{k}}_y\alpha$$

where $\hat{\mathfrak{k}}_y$ denotes the class sum of y. Since $d(y) = D$ and $\mathcal{I}(D)$ is an ideal of $\mathfrak{Z}$ we have $\beta^* \in \mathcal{I}(D)$.

Consider β_j. By Lemma 1(iii), we can write $\beta_j = \Sigma c_u u$ where all $u \approx g_j$, for some fixed g_j, and where $\Sigma c_u = 0$. Thus $\beta_j^* = \Sigma c_u u^*$. Now, since D is central, the condition $u \approx g_j$ yields easily

$$u^* = d_u g_j^* = d_u[\mathbf{C}(g_j) : D]\hat{\mathfrak{k}}_{g_j}$$

for some element $d_u \in D$. Thus

$$\beta_j^* = \sum c_u u^* = \left(\sum c_u d_u\right)[\mathbf{C}(g_j) : D]\hat{\mathfrak{k}}_{g_j} \in \omega(K[D]) \cdot \mathcal{I}(D) \subseteq \mathcal{I}(D)$$

since $\Sigma c_u = 0$ and $d(g_j) = D$, by definition of β_j.

Now $\beta^* = \beta_1^* + \cdots + \beta_s^* + \gamma^*$ and we know that $\beta^*, \beta_j^* \in \mathcal{I}(D)$. Hence $\gamma^* \in \mathcal{I}(D)$. But γ involves only group elements with defect group strictly larger than D, so this yields $\gamma^* = 0$. Hence

$$[\mathbf{C}(y) : D]\hat{\mathfrak{k}}_y\alpha = \beta^* = \beta_1^* + \beta_2^* + \cdots + \beta_s^* \in \omega(K[D]) \cdot \mathcal{I}(D)$$

and, since $[\mathbf{C}(y): D]$ is a p'-number, we have $\hat{\mathfrak{k}}_y\alpha \in \omega(K[D]) \cdot \mathcal{I}(D)$. This clearly proves part (i).

(ii) Since $\mathcal{I}(D) \cdot J\mathfrak{Z} = \omega(K[D]) \cdot \mathcal{I}(D)$, it follows easily by induction that $\mathcal{I}(D) \cdot (J\mathfrak{Z})^i = \omega(K[D])^i \cdot \mathcal{I}(D)$. Since $\omega(K[D])^{p^d} = 0$, the result follows.

Lemma 3. *Let D be any p-subgroup of G with $|D| = p^d$. Then*
(i) $\mathcal{I}(D) \cdot (J\mathfrak{Z})^{p^d} \subseteq \tilde{\mathcal{I}}(D)$,
(ii) $\mathcal{I}(D) \cdot (J\mathfrak{Z})^{(p^{d+1}-1)/(p-1)} = 0.$

Proof. (i) Since $\mathcal{I}(D)$ is an ideal of $\mathfrak{Z}$, we have $\mathcal{I}(D) \cdot (J\mathfrak{Z})^{p^d} \subseteq \mathcal{I}(D)$. Thus we need only show that the left-hand term has no class sums with defect group D. But observe that if $\hat{\mathfrak{k}}$ is a class sum with defect group D, then $\pi(\hat{\mathfrak{k}}) \neq 0$ where $\pi = \pi_{\mathbf{C}(D)}$ denotes the Brauer homomorphism from $\mathfrak{Z}$ to $\mathbf{Z}(K[\mathbf{C}(D)])$. Thus it clearly suffices to show that

$$\pi(\mathcal{I}(D)) \cdot \pi(J\mathfrak{Z})^{p^d} = 0.$$

Let $C = \mathbf{C}(D)$. Then by definition, $\pi(\mathfrak{Z}) \subseteq \bar{\mathfrak{Z}} = \mathbf{Z}(K[C])$, and, since nilpotent elements map to nilpotent elements, we have $\pi(J\mathfrak{Z}) \subseteq J\bar{\mathfrak{Z}}$. Now let g be a support element in $\pi(\mathcal{I}(D))$. Since $g \in C$ and $\hat{\mathfrak{k}}_g \in \mathcal{I}(D)$, it follows that D is a Sylow p-subgroup of $\mathbf{C}_G(g)$. Furthermore, if D_1 is a Sylow p-subgroup of $\mathbf{C}_C(g) = C \cap \mathbf{C}_G(g)$, then, since $D_1 \subseteq C$, we see that D_1D is a p-subgroup of $\mathbf{C}_G(g)$ so $D_1D = D$ and $D_1 \subseteq D \cap C = \bar{D}$, the center of D. We conclude from all of this that

$$\pi(\mathcal{I}(D)) \subseteq \mathcal{I}_C(\bar{D})$$

where $\mathcal{I}_C(\bar{D})$ denotes the span of all class sums in $K[C]$ with defect group contained in $\bar{D}$, a central p-subgroup of C. Thus

$$\pi(\mathfrak{I}(D)) \cdot \pi(J\mathfrak{Z})^{p^d} \subseteq \mathfrak{I}_C(\bar{D}) \cdot (J\bar{\mathfrak{Z}})^{p^d}$$

and the latter is zero, by Lemma 2(ii), since $\bar{D}$ is central in C and $|\bar{D}| < |D| = p^d$. In view of our previous remarks, this proves (i).

(ii) We proceed by induction on d. The case $d = 0$ follows from Lemma 2(ii). If $D \neq \langle 1 \rangle$, then

$$\tilde{\mathfrak{I}}(D) = \sum_{D_i < D} \mathfrak{I}(D_i)$$

so by induction

$$\tilde{\mathfrak{I}}(D) \cdot (J\mathfrak{Z})^{(p^d-1)/(p-1)} = 0.$$

Hence since $\mathfrak{I}(D) \cdot (J\mathfrak{Z})^{p^d} \subseteq \tilde{\mathfrak{I}}(D)$, by (i) above, we have clearly $\mathfrak{I}(D) \cdot (J\mathfrak{Z})^f = 0$ where

$$f = p^d + (p^d - 1)/(p - 1) = (p^{d+1} - 1)/(p - 1).$$

Proof of the Theorem. Let $|G| = p^a b$ and let P be a Sylow p-subgroup of G. Then $|P| = p^a$ and $1 \in \mathfrak{I}(P)$ so Lemma 3(ii), with $D = P$, yields

$$(J\mathfrak{Z})^{(p^{a+1}-1)/(p-1)} = 0.$$

We close with two brief remarks. First, if $G = \langle x \rangle$ is cyclic of order p^a, then $J\mathfrak{Z} = \omega(K[G])$ and the nilpotence degree of $J\mathfrak{Z}$ is precisely p^a. Thus the bound given in the main theorem cannot be significantly decreased without additional assumptions on the structure of P.

Second, unfortunately a bound on the nilpotence degree of $J\mathfrak{Z}$ does not yield a bound on the nilpotence degree of $JK[G]$. For example, let G be an extra special p-group of order p^{2n+1}. Then $\mathfrak{Z} = K[Z] + \hat{Z} \cdot K[G]$ where Z is the center of G, so $J\mathfrak{Z} = \omega(K[Z]) + \hat{Z} \cdot K[G]$ and $(J\mathfrak{Z})^p = 0$. On the other hand, $JK[G] = \omega(K[G])$ so the nilpotence degree of $JK[G]$ is easily seen, using [**1**, Theorems 3.3.6 and 11.1.9], to be $p + 2n(p - 1)$. Thus the nilpotence degree of $JK[G]$ can be arbitrarily large while the nilpotence degree of $J\mathfrak{Z}$ remains equal to p.

Acknowledgement. Research partially supported by NSF Grant No. MCS 77-01775 A01. This work was done during a brief visit to the University of Essen in May 1978. I would like to thank Professor Gerhard Michler and Dr. Martin Lorenz for their kind hospitality during that stay.

References

1. D. S. Passman, *The algebraic structure of group rings*, Wiley-Interscience, New York, 1977.
2. Wolfgang Willems, *Über die Existenz von Blöcken*, J. Algebra **53** (1978), 402–409.

Department of Mathematics, University of Wisconsin-Madison, Madison, Wisconsin 53706