ON BOUNDEDNESS OF INTEGRABLE AUTOMORPHIC FORMS IN $\mathbb{C}^n$

SU-SHING CHEN

Abstract. We give a necessary and sufficient condition for an integrable automorphic form on a bounded symmetric domain $D$ in $\mathbb{C}^n$ to be bounded.

The question on the boundedness of integrable automorphic forms on the unit disk in $\mathbb{C}^1$ has been investigated in [1], [5], [6], [7], [8], [9] and many other papers. Integrable automorphic forms on the bounded homogeneous domains in $\mathbb{C}^n$ have been considered by Earle [2] and Selberg [10]. In this paper, we shall study the boundedness of integrable automorphic forms on the bounded symmetric domains in $\mathbb{C}^n$.

Let $D$ be a bounded symmetric domain in $\mathbb{C}^n$ with Bergman kernel function $k(z, w)$, where $z$ and $w$ represent $n$-tuples $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ respectively.

For every holomorphic automorphism $g$ of $\text{Aut}(D)$, we have $k(gz, gw) = k(g(z), g(w))$, where $g'(z)$ is the complex Jacobian of the automorphism $g$. The volume element $dm(z) = k(z, z)dz$ is invariant under the group $\text{Aut}(D)$ of all holomorphic automorphisms of $D$, where $dz$ is the euclidean volume element of $D$.

Let $\Gamma$ be a discrete subgroup of $\text{Aut}(D)$. We choose a fundamental domain $R$ for $\Gamma$ so that $\partial R \cap D$ has zero volume. A function $f$ holomorphic on $D$ is said to be an automorphic form of dimension $-2q$ if $f(gz)g'(z)^q = f(z)$ for all $z$ in $D$ and $g$ in $\Gamma$. We denote by $A_q(\Gamma)$ the space of integrable forms, i.e., the set of all holomorphic automorphic forms $f$ of dimension $-2q$ such that

$$\|f\|_q = \int_R |f(z)| |k(z, z)|^{-q/2}dm(z) < \infty.$$ We denote by $B_q(\Gamma)$ the space of bounded forms, i.e., the set of all forms of dimension $-2q$ such that

$$\|f\|_\infty = \sup_{z \in D} |f(z)k(z, z)|^{-q/2} < \infty.$$ We refer to the paper [2] of Earle for notations and basic facts. In particular, $q$ is any integer $> 2$ so that all formulas in [2] are valid. $c(q)$ is a certain constant depending only on $q$.

The following theorem is a generalization of a result of Metzger and Rao [7].
THEOREM. Let $D$ be a bounded symmetric domain in $\mathbb{C}^n$ and let $\Gamma$ be a discrete subgroup of $\text{Aut}(D)$. For $q > 2$, $A_q(\Gamma) \subset B_q(\Gamma)$ if and only if

$$\sup_{z \in D} |k(z, z)^{-q}\alpha(z, z)| < \infty,$$

where $\alpha(z, w) = c(q)\sum_{\gamma \in \Gamma} k(\gamma z, \gamma z)^q \gamma^q(z)^q$.

PROOF. According to Theorem 3.1, Corollary 5.2 and Theorem 7 of [2], the function $\alpha(z, w)$, for a fixed $w$ in $D$, belongs to $A_q(\Gamma)$ and $B_q(\Gamma)$.

If $f$ is in $A_q(\Gamma)$, then by [2], we have

$$f(w) = c(q)\int_D f(z)k(w, z)^q k(z, z)^{-q/2} dm(z)$$

$$= c(q)\int \sum_{\gamma \in \Gamma} f(\gamma z)k(w, \gamma z)^q k(\gamma z, \gamma z)^{-q/2} dm(z)$$

$$= c(q)\int \sum_{\gamma \in \Gamma} f(\gamma z)k(w, \gamma z)^q k(z, z)^{-q/2} \gamma(z)^q dm(z)$$

$$= c(q)\int \sum_{\gamma \in \Gamma} k(w, \gamma z)^q \gamma(z)^q f(z)k(z, z)^{-q/2} dm(z)$$

$$= c(q)\int f(z) \alpha(z, w) k(z, z)^{-q/2} dm(z).$$

Consequently,

$$|f(w)| |k(w, w)^{-q/2}| < c(q)\|f\|_q \sup_{z, w \in D} |\alpha(z, w)| |k(w, w)|^{-q/2}|k(z, z)|^{-q/2}.$$

Since $\alpha(z, w)$ is in $A_q(\Gamma)$ for a fixed $w$ in $D$,

$$\alpha(w, w) = c(q)\int |\alpha(z, w)|^2 k(z, z)^{-q} dm(z)$$

and

$$\alpha(z, w) = c(q)\int \alpha(w', w) \alpha(w', z) k(w', w')^{-q} dm(w').$$

The Schwarz inequality implies that $|\alpha(z, w)|^2 < |\alpha(z, z)\alpha(w, w)|$. Thus $f$ is in $B_q(\Gamma)$.

Conversely, by the closed graph theorem, we have a bounded linear map from $A_q(\Gamma)$ into $B_q(\Gamma)$. Thus, there exists a positive constant $C$ such that for all $f$ in $A_q(\Gamma)$ and $z$ in $D$

$$|f(z)| |k(z, z)^{-q/2}| < C\|f\|_q.$$

For the function $\alpha(\cdot, w)$, and $z$, in $D$,

$$|\alpha(z, w)| |k(z, z)^{-q/2}| < C\|\alpha(\cdot, w)\|_q$$

and

$$|\alpha(w, w)| |k(w, w)^{-q/2}| < C\|\alpha(\cdot, w)\|_q.$$
But

\[ \|\alpha(\cdot, w)\|_q \leq c(q) \int_R \sum_{\gamma \in \Gamma} |k(\gamma z, w)|^q |\gamma'(z)|^q |k(z, z)^{-q/2}| \ dm(z) \]

\[ = c(q) \int_D |k(z, w)|^q |k(z, z)^{-q/2}| \ dm(z) \]

\[ = c(q)/c(q/2)|k(w, w)^{q/2}| \]

by a formula in [2]. Thus (***) is satisfied.

REFERENCES


Department of Mathematics, University of Florida, Gainesville, Florida 32611