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DIEUDONNÉ-SCHWARTZ THEOREM ON BOUNDED SETS IN INDUCTIVE LIMITS

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ABSTRACT. The Dieudonné-Schwartz Theorem for bounded sets in strict inductive limits does not hold for general inductive limits $E = \operatorname{ind}\lim E_n$. It does if every closed convex set in E_n is closed in E_{n+1}. This condition is not necessary. In case all spaces E_n are normed a necessary and sufficient condition for the validity of the Dieudonné-Schwartz Theorem is given.

Let $E_1 \subset E_2 \subset \cdots$ be a sequence of locally convex spaces and $E = \operatorname{ind}\lim E_n$ their inductive limit (with respect to the identity maps id: $E_n \to E_{n+1}$). The Dieudonné-Schwartz Theorem (further referred to as DST), see [2, Chapter 2, §12], states that a set $B \subset E$ is bounded if and only if it is contained and bounded in some E_n, provided that

(H-1) each E_n is closed in E_{n+1} and

(H-2) the topology of each E_n equals the topology induced in E_n by E_{n+1}.

These two hypotheses imply [2, Chapter 2, §12]

(H-3) each E_n is closed in E.

It is shown in [3] that if H-3 holds and B is a bounded set in E, then $B \subset E_n$ for some n, but may not be bounded there. Therefore, in order to preserve the DST, we need a stronger hypothesis than H-3. We introduce three more.

(H-4) each convex and closed set in E_n is closed in E_{n+1},

(H-5) for each set B bounded and convex in E_n, the closure $\bar{B}^E$ of B in E is contained and bounded in E_{n+p} for some $p \in N$,

(H-6) for each set B bounded and convex in E_n, the closure $\bar{B}^E$ of B in E is contained in E_{n+p} for some $p \in N$.

LEMMA 1. H-4 $\Rightarrow$ H-3.

PROOF. Assume that E_1 is not closed in E and $x \in \bar{E}_1^E \setminus E_1$. Since E_1 is closed in E_2, there exists a closed convex neighborhood U_2 of 0 in E_2 such that $x \notin E_1 + 2U_2$. Now, $\overline{E_1 + U_2}^E$ is closed convex in E_2 and, by H-4, closed in E_3. Since $\overline{E_1 + U_2}^E \subset E_1 + 2U_2$, there exists a closed convex neighborhood U_3 of 0 in E_3 such that $x \notin E_1 + U_2 + 2U_3$. When all $U_2, U_3, \ldots$ are constructed, the set $E_1 + \bigcup_{k=2}^{\infty}(U_2 + U_3 + \cdots + U_k)$ is a neighborhood of E_1 in E which does not contain x, a contradiction.

THEOREM 1. H-4 $\Rightarrow$ *DST*.

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PROOF. Let B be a bounded set in E. According to Lemma 1 and [3], $B \subset E_n$ for some n. Put $n = 1$ and assume B is not bounded in any E_m, $m \in N$.

Since E_1 is a locally convex space, B is not weakly bounded there and there exists a continuous linear functional $f_1: E_1 \to R$ which is unbounded on B. Choose a sequence $\{b_k\} \subset B$ such that $f_1(b_k) > k$, $k = 1, 2, \ldots$. The set $U_1 = \{x \in E_1; f_1(x) \leqslant 1\}$ is closed convex in E_1, hence closed in E_2, and there exists a continuous linear functional $g: E_2 \to R$ such that $U_1 \subset \{x \in E_2; g(x) < 1\}$ and $g(b_1) > 1$.

If $f_1(x) = 0$, then $f_1(kx) = 0$ for every integer k and $kx \in U_1$. This implies $g(kx) = 0$ and $g(x) = 0$. Hence $g|_{E_1} = cf_1$, where $g|_{E_1}$ is the restriction of g to E_1. Then $f_2 = g/c$ is a continuous extension of f_1 to E_2. The set $U_2 = \{x \in E_2; f_2(x) < 1\}$ is a closed convex neighborhood of 0 in E_2 for which $U_1 \subset U_2$ and $b_1 \notin U_2, b_2/2 \notin U_2$.

Since U_2 is closed in E_3, the process can be repeated until we get a sequence $\{f_k: E_k \to R; k = 1, 2, 3, \ldots\}$ of continuous linear functionals, each of which is an extension of its predecessor, and $b_r/r \notin U_k = f_k^{-1}(-\infty, 1]$ for $r = 1, 2, \ldots, k$. The set $U = \bigcup_{k=1}^{\infty} U_k$ is a neighborhood of 0 in E and $B \subset sU$ for some $s \in N$. But $b_s/s \notin U$, which is a contradiction.

THEOREM 2. *If all E_n are normed spaces, then* H-5 *is equivalent to DST.*

PROOF. 1. Let DST hold and B be bounded and convex in E_n. Then B and $\bar{B}^E$ are bounded in E and $\bar{B}^E$ must be bounded in some E_{n+p}. We did not need normability of the E_n's.

2. Let H-5 hold and B be bounded in E but not bounded in any E_n. Denote by B_n the closed unit ball in E_n. There exists $b_1 \in B \setminus \{0\}$ and a closed convex neighborhood V_1 of 0 in E such that $b_1 \notin V_1$. For some $p_1 \in N$, $b_1 \in E_{p_1}$. Put $U_1 = \overline{V_1 \cap B_{p_1}}^E$. Then $U_1 \subset V_1$ and $b_1 \notin U_1$. Since $V_1 \cap B_{p_1}$ is bounded and convex in E_{p_1}, U_1 is contained and bounded in some E_{p_2}. Hence there exists $b_2 \in B \setminus 2U_1$. We may take p_2 so that $p_2 > p_1$ and $b_2 \in E_{p_2}$. Further, U_1 is closed and convex in E. Hence there exists a closed convex neighborhood V_2 of 0 in E such that b_1, $b_2/2 \notin U_1 + 2V_2$. Put $U_2 = \overline{V_2 \cap B_{p_2}}^E$. Again, $U_2 \subset V_2$ and b_1, $b_2/2 \notin \overline{U_1 + U_2}^E \subset \overline{U_1 + V_2}^E \subset U_1 + V_2 + V_2$.

We repeat this process until we get sequences $\{b_k\} \subset B$, $p_1 < p_2 < \cdots$, and a sequence of closed convex neighborhoods $V_1, V_2, \ldots$ of 0 in E, such that $b_k/k \notin U_1 + U_2 + \cdots + U_n$ for $k = 1, 2, \ldots, n$, where $U_k = \overline{V_k \cap B_{p_k}}^E$. Then $U = \bigcup_{k=1}^{\infty}(U_1 + U_2 + \cdots + U_k)$ is a neighborhood of 0 in E and $B \subset sU$ for some s. But $b_s/s \notin U$, a contradiction.

With a slight modification of the last proof we can get

THEOREM 3. *If all E_n are normal spaces then* H-6 *is equivalent to*: *Each bounded set in E is contained in some E_n.*

LEMMA 2. *Let X, Y be Banach spaces, $X \subset Y$,* id: *$X \to Y$ continuous, and X reflexive. Then every bounded closed convex set in X is closed in Y.*

The proof follows from the Alaoglu Theorem.

THEOREM 4. *If all E_n are reflexive Banach spaces, then DST holds.*

PROOF. It is sufficient to show that H-5 holds. Let B be a bounded closed convex set in E_n and $b \notin B$. There exists a bounded closed convex neighborhood U_0 of 0 in E_n such that $b \notin B + U_0$. By Lemma 2, $B + U_0$ is bounded closed and convex in E_{n+1}. Hence there is a bounded closed convex neighborhood U_1 of 0 in E_{n+1} such that $b \notin B + U_0 + U_1$, etc. The set $U = \bigcup_{k=0}^{\infty}(U_0 + U_1 + \cdots + U_k)$ is a neighborhood of 0 in E and $b \notin B + U$, i.e. B is closed in E.

EXAMPLE. Let $R_+ = [0, \infty)$, $w_n(x) = \exp x/n$, $x \geqslant 0$, $E_n = \{f \in L^2(R_+); \|w_n f\|_2 < +\infty\}$, $n \in N$. All E_n are Hilbert spaces with the inner product $(f, g) \mapsto \langle w_n f, w_n g\rangle_2$, $E_1 \subset E_2 \subset \cdots$, and id: $E_n \to E_{n+1}$ are continuous. By Theorem 4, DST holds. We show that H-1, and hence both H-3 and H-4, do not hold. It means that H-4 is not a necessary condition in Theorem 1.

Take $n \in N$ and $1/(n+1) < a < b < 1/n$. Then $\exp(-ax) \in E_{n+1} \setminus E$. The functions

$$f_k(x) = \begin{cases} \exp(-ax) & \text{for } 0 \leqslant x \leqslant k, \\ \exp(-bx) & \text{for } k \leqslant x, \end{cases}$$

all belong to E_n and converge in E_{n+1} to $\exp(-ax)$.

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