DIEUDONNÉ-SCHWARTZ THEOREM ON BOUNDED SETS IN INDUCTIVE LIMITS

J. KUCERA AND K. McKENNON

Abstract. The Dieudonné-Schwartz Theorem for bounded sets in strict inductive limits does not hold for general inductive limits $E = \text{ind lim } E_n$. It does if every closed convex set in $E_n$ is closed in $E_{n+1}$. This condition is not necessary. In case all spaces $E_n$ are normed a necessary and sufficient condition for the validity of the Dieudonné-Schwartz Theorem is given.

Let $E_1 \subset E_2 \subset \cdots$ be a sequence of locally convex spaces and $E = \text{ind lim } E_n$ their inductive limit (with respect to the identity maps $\text{id}: E_n \rightarrow E_{n+1}$). The Dieudonné-Schwartz Theorem (further referred to as DST), see [2, Chapter 2, §12], states that a set $B \subset E$ is bounded if and only if it is contained and bounded in some $E_n$, provided that

(H-1) each $E_n$ is closed in $E_{n+1}$ and
(H-2) the topology of each $E_n$ equals the topology induced in $E_n$ by $E_{n+1}$.

These two hypotheses imply [2, Chapter 2, §12]

(H-3) each $E_n$ is closed in $E$.

It is shown in [3] that if H-3 holds and $B$ is a bounded set in $E$, then $B \subset E_n$ for some $n$, but may not be bounded there. Therefore, in order to preserve the DST, we need a stronger hypothesis than H-3. We introduce three more.

(H-4) each convex and closed set in $E_n$ is closed in $E_{n+1}$,

(H-5) for each set $B$ bounded and convex in $E_n$, the closure $\overline{B}^E$ of $B$ in $E$ is contained and bounded in $E_{n+p}$ for some $p \in \mathbb{N}$,

(H-6) for each set $B$ bounded and convex in $E_n$, the closure $\overline{B}^E$ of $B$ in $E$ is contained in $E_{n+p}$ for some $p \in \mathbb{N}$.

Lemma 1. H-4 $\Rightarrow$ H-3.

Proof. Assume that $E_1$ is not closed in $E$ and $x \in \overline{E_1}^E \setminus E_1$. Since $E_1$ is closed in $E_2$, there exists a closed convex neighborhood $U_2$ of 0 in $E_2$ such that $x \notin E_1 + 2U_2$. Now, $\overline{E_1 + U_2}^E$ is closed convex in $E_2$ and, by H-4, closed in $E_3$. Since $\overline{E_1 + U_2}^E \subset E_1 + 2U_2$, there exists a closed convex neighborhood $U_3$ of 0 in $E_3$ such that $x \notin E_1 + U_2 + 2U_3$. When all $U_2, U_3, \ldots$ are constructed, the set $E_1 + \bigcup_{k=2}^{\infty} (U_2 + U_3 + \cdots + U_k)$ is a neighborhood of $E_1$ in $E$ which does not contain $x$, a contradiction.

Theorem 1. H-4 $\Rightarrow$ DST.
Proof. Let $B$ be a bounded set in $E$. According to Lemma 1 and [3], $B \subset E_n$ for some $n$. Put $n = 1$ and assume $B$ is not bounded in any $E_m$, $m \in \mathbb{N}$.

Since $E_1$ is a locally convex space, $B$ is not weakly bounded there and there exists a continuous linear functional $f_1: E_1 \to \mathbb{R}$ which is unbounded on $B$. Choose a sequence $\{b_k\} \subset B$ such that $f_1(b_k) > k$, $k = 1, 2, \ldots$. The set $U_1 = \{x \in E_1; f_1(x) < 1\}$ is closed convex in $E_1$, hence closed in $E_2$, and there exists a continuous linear functional $g: E_2 \to \mathbb{R}$ such that $U_1 \subset \{x \in E_2; g(x) < 1\}$ and $g(b_1) > 1$.

If $f_1(x) = 0$, then $f_1(kx) = 0$ for every integer $k$ and $kx \in U_1$. This implies $g(kx) = 0$ and $g(x) = 0$. Hence $g|_{E_1} = c$, where $g|_{E_1}$ is the restriction of $g$ to $E_1$. Then $f_2 = g/c$ is a continuous extension of $f_1$ to $E_2$. The set $U_2 = \{x \in E_2; f_2(x) < 1\}$ is a closed convex neighborhood of $0$ in $E_2$ for which $U_1 \subset U_2$ and $b_1 \notin U_2, b_2/2 \notin U_2$.

Since $U_2$ is closed in $E_2$, the process can be repeated until we get a sequence $\{f_k: E_k \to \mathbb{R}; k = 1, 2, 3, \ldots \}$ of continuous linear functionals, each of which is an extension of its predecessor, and $b_r/r \notin U_k = f_k^{-1}(-\infty, 1]$ for $r = 1, 2, \ldots, k$.

The set $U = \bigcup_{k=1}^{\infty} U_k$ is a neighborhood of $0$ in $E$ and $B \subset sU$ for some $s \in N$. But $b_r/s \notin U$, which is a contradiction.

Theorem 2. If all $E_n$ are normed spaces, then H-5 is equivalent to DST.

Proof. 1. Let DST hold and $B$ be bounded and convex in $E_n$. Then $B$ and $\overline{B}^E$ are bounded in $E$ and $\overline{B}^E$ must be bounded in some $E_{n+p}$. We did not need normability of the $E_n$'s.

2. Let H-5 hold and $B$ be bounded in $E$ but not bounded in any $E_n$. Denote by $B_n$ the closed unit ball in $E_n$. There exists $b_1 \in B^\sim \{0\}$ and a closed convex neighborhood $V_1$ of $0$ in $E$ such that $b_1 \notin V_1$. For some $p_1 \in N$, $b_1 \in E_{p_1}$. Put $U_1 = V_1 \cap B_{p_1}^E$. Then $U_1 \subset V_1$ and $b_1 \notin U_1$. Since $V_1 \cap B_{p_1}$ is bounded and convex in $E_{p_1}$, $U_1$ is contained and bounded in some $E_{p_2}$. Hence there exists $b_2 \in B\setminus 2U_1$. We may take $p_2$ so that $p_2 > p_1$ and $b_2 \in E_{p_2}$. Further, $U_1$ is closed and convex in $E$. Hence there exists a closed convex neighborhood $V_2$ of $0$ in $E$ such that $b_1, b_2/2 \notin U_1 + 2V_2$. Put $U_2 = V_2 \cap B_{p_2}^E$. Again, $U_2 \subset V_2$ and $b_1, b_2/2 \notin U_1 + U_2 \subset U_1 + V_2$.

We repeat this process until we get sequences $\{b_k\} \subset B, p_1 < p_2 < \ldots$, and a sequence of closed convex neighborhoods $V_1, V_2, \ldots$ of $0$ in $E$, such that $b_k/\k \notin U_1 + U_2 + \cdots + U_n$ for $k = 1, 2, \ldots, n$, where $U_k = V_k \cap B_{p_k}^E$. Then $U = \bigcup_{k=1}^{\infty} (U_1 + U_2 + \cdots + U_k)$ is a neighborhood of $0$ in $E$ and $B \subset sU$ for some $s$. But $b_r/s \notin U$, which is a contradiction.

With a slight modification of the last proof we can get

Theorem 3. If all $E_n$ are normal spaces then H-6 is equivalent to: Each bounded set in $E$ is contained in some $E_n$.

Lemma 2. Let $X, Y$ be Banach spaces, $X \subset Y$, id: $X \to Y$ continuous, and $X$ reflexive. Then every bounded closed convex set in $X$ is closed in $Y$.

The proof follows from the Alaoglu Theorem.

Theorem 4. If all $E_n$ are reflexive Banach spaces, then DST holds.
Proof. It is sufficient to show that H-5 holds. Let $B$ be a bounded closed convex set in $E_n$ and $b \notin B$. There exists a bounded closed convex neighborhood $U_0$ of 0 in $E_n$ such that $b \notin B + U_0$. By Lemma 2, $B + U_0$ is bounded closed and convex in $E_{n+1}$. Hence there is a bounded closed convex neighborhood $U_1$ of 0 in $E_{n+1}$ such that $b \notin B + U_0 + U_1$, etc. The set $U = \bigcup_{k=0}^{\infty}(U_0 + U_1 + \cdots + U_k)$ is a neighborhood of 0 in $E$ and $b \notin B + U$, i.e. $B$ is closed in $E$.

Example. Let $R_+ = [0, \infty)$, $w_n(x) = \exp x/n$, $x > 0$, $E_n = \{f \in L^2(R_+); \|w_n f\|_2 < +\infty\}$, $n \in \mathbb{N}$. All $E_n$ are Hilbert spaces with the inner product $(f, g) \mapsto \langle w_n f, w_n g \rangle_2$, $E_1 \subseteq E_2 \subseteq \cdots$, and id: $E_n \to E_{n+1}$ are continuous. By Theorem 4, DST holds. We show that H-1, and hence both H-3 and H-4, do not hold. It means that H-4 is not a necessary condition in Theorem 1.

Take $n \in \mathbb{N}$ and $1/(n+1) < a < b < 1/n$. Then $\exp(-ax) \in E_{n+1} \setminus E$. The functions

$$f_k(x) = \begin{cases} \exp(-ax) & \text{for } 0 < x < k, \\ \exp(-bx) & \text{for } k < x, \end{cases}$$

all belong to $E_n$ and converge in $E_{n+1}$ to $\exp(-ax)$.

References


Department of Mathematics, Washington State University, Pullman, Washington 99164

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use