

A SIMPLICITY THEOREM FOR AMOEBAS OVER RANDOM REALS

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ABSTRACT. Let M be a countable standard transitive model of ZFC, \mathcal{Q} be an amoeba over M , and r be a random real over M .

THEOREM. (a) *There is no infinite set of reals X contained in the complement of \mathcal{Q} with $X \in M[r]$; (b) If $\{x_n \mid n < \omega\} \in M[r]$ is a sequence of distinct reals, then for all large enough n , $\{x_i \mid 2^n < i < 2^{n+1}\} \cap \mathcal{Q} \neq \emptyset$.*

Introduction. Solovay's by now classical construction² of a model of set theory in which there is no non-Lebesgue-measurable set of reals has prompted a number of unsuccessful attempts to obtain a similar model without recourse to any large cardinal hypothesis. There is a certain plausible approach motivated by the observation that iterated amoeba forcing, à la Martin's axiom, produces a model in which, for any real p , almost every real is random over $L(p)$.³ The stumbling block here has been the failure to obtain a "factoring" lemma, i.e., to analyze the structure of what happens to the forcing algebras involved when they are reduced over a given random real. The basic problem would be to factor an amoeba over a random real.

Part (a) of the Theorem stated in the Abstract says that for r random over L , \mathcal{Q} an amoeba over L , and $r \in L[\mathcal{Q}]$, the information about \mathcal{Q} encoded in r , or, more specifically, the information about \mathcal{Q} 's complement, is severely limited. \mathcal{Q} , of course, contains every real in L . Part (b) gives a certain sense in which \mathcal{Q} contains a thick portion of the reals in $L[r]$ also. (The function $n \mapsto 2^n$ is not a tight bound. The point is that there is a certain thickness obtained independent of the choice of \mathcal{Q} , r , and the sequence of reals.)

Aside from usual forcing technology and absoluteness considerations, the main instrument of the proof is a combinatorial, measure-theoretic lemma about "covering" sequences of measurable functions with sets of small measure. Before getting into the proofs themselves, let us make the notions involved precise.

Terminology and notation. By the "real numbers" we mean Cantor space, i.e., 2^ω endowed with the usual topology. It will sometimes be useful to make 2^ω correspond with the usual middle-third Cantor set. To this end, define $\nu(x) = \sum_{x(i)=1} 2 \cdot 3^{-(i+1)}$ for $x \in 2^\omega$. By "Lebesgue measure" on the reals we mean the usual

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²See Solovay [1970].

³For general background on iterated forcing and Martin's axiom, see Martin-Solovay [1970].

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probability measure μ such that for each $i < \omega$, $\mu(\{x|x(i) = 0\}) = \mu(\{x|x(i) = 1\}) = \frac{1}{2}$.

For $h: \omega \rightarrow$ the reals of ordinary real analysis and $A \subseteq \omega$, we say that A is *h-thick* if for all large enough n , $\{i \in \omega|h(n) \leq i < h(n + 1)\} \cap A \neq \emptyset$.

Let $\bar{\epsilon}$ be a fixed rational, $0 < \bar{\epsilon} < 1$. $\mathcal{P}_{\bar{\epsilon}}$ is $\{X \subseteq 2^\omega|X \text{ open and } \mu(x) < \bar{\epsilon}\}$, partially ordered so that X is stronger than Y iff $X \supseteq Y$. \mathcal{S} is $\{X \subseteq 2^\omega|X \text{ closed and } \mu(X) > 0\}$, partially ordered so that X is stronger than Y iff $X \subseteq Y$. For the remainder of this paper, let M be a fixed countable standard transitive model of ZFC. \mathcal{Q} is an amoeba over M means that \mathcal{Q} is M -generic on $(\mathcal{P}_{\bar{\epsilon}})^M$. \mathcal{Q} is freely thought of either as an open set of reals of measure $\bar{\epsilon}$, or as a code for such a set, and for x any real (not necessarily in $M[\mathcal{Q}]$), the meaning of " $x \in \mathcal{Q}$ " is well defined and absolute. Simple density arguments show that \mathcal{Q} contains every set of measure 0 coded in M . r is M -random means that r is the real determined by an M -generic object on $(\mathcal{S})^M$. r is M -random iff r lies in every set of reals of measure 1 coded in M .

For the Theorem stated above to apply nontrivially to factoring an amoeba over a random real, it would have to be the case that in $M[\mathcal{Q}]$ there is a random real r such that $(2^\omega \cap M[r]) - \mathcal{Q}$ is infinite. Observation 1 below shows that this is always the case.

Notation. For any real r and $n < \omega$, $r \oplus n$ is the real given by $(r \oplus n)(i) = r(i)$ iff $i \neq n$.

Observation 1. Let \mathcal{Q} be an amoeba over M . Then in $M[\mathcal{Q}]$ there is a real r such that $\{n|r \oplus n \notin \mathcal{Q}\}$ is infinite, and any such real is M -random.

PROOF. Work in $M[\mathcal{Q}]$. Let $X = \{r|\{n|r \oplus n \notin \mathcal{Q}\} \text{ is infinite}\}$. We show, in fact, that $\mu(X) > 0$. Let $Y_m = \{x|\forall n > m (r \oplus n \in \mathcal{Q})\}$. $2^\omega - X = \cup_{m < \omega} Y_m$. If $\mu(X) = 0$ then let m be such that $\mu(Y_m) > \bar{\epsilon}$. Now, for all $x \in Y_m$, $x \oplus (m + 1) \in \mathcal{Q}$, and it is routine to see that $\mu(\{x \oplus (m + 1)|x \in Y_m\}) = \mu(Y_m)$, so $\mu(\mathcal{Q}) > \bar{\epsilon}$, a contradiction.

Let $r \in X$, and take n such that $r \oplus n \notin \mathcal{Q}$. Then $r \oplus n$ is M -random as it can be in no set of measure 0 coded in M , each of these being contained in \mathcal{Q} . It is easy to check that as $r \oplus n$ is M -random, so must be r . \square

Observation 1 gives us an example of an infinite sequence of reals in $M[r]$, namely $\{x_n = r \oplus n|n < \omega\}$, such that $\{x_n\} - \mathcal{Q}$ is infinite but, by the Theorem we shall prove, it has no infinite subset in $M[r]$.

The proof of the Theorem proceeds by giving an appropriate representation of reals in $M[r]$ as Borel functions in M and then by proving a "covering" principle for sequences of measurable functions. As there will be several different models of set theory involved, it will be necessary to speak of "codes" for Borel functions. Rather than going into details, suffice it to say that we fix a way of representing (total) Borel functions: $2^\omega \rightarrow 2^\omega$ by elements of 2^ω in such a way that the predicate $C(x) \equiv$ " x is a code for a total Borel function" is Π_1^1 and the predicate $F(x, y, z) \equiv$ " x is a code for a total Borel function applied to argument y yields value z " is Δ_1^1 in the sense that there are Σ_1^1 and Π_1^1 predicates P and Q such that whenever $C(x)$ holds then for all y and z , $F(x, y, z) \leftrightarrow P(x, y, z) \leftrightarrow Q(x, y, z)$. Thus C and F are absolute

with respect to standard transitive models of ZF.

Notation. For \mathcal{P} a (description of) a partial ordering, we write $p \Vdash_{\mathcal{P}} \mathcal{F}$ to mean that $M \models p \in \mathcal{P}$ and p weakly forces \mathcal{F} over \mathcal{P} .

PROPOSITION 2. *Suppose $X \Vdash_{\mathcal{S}} t \in 2^\omega$. Then there is $x \in M$ such that $C(x)$ and $X \Vdash_{\mathcal{S}} F(x, \mathbf{r}, t)$, where \mathbf{r} is the canonical name for the real given by the generic object on \mathcal{S} .*

PROOF. Work in M . For each $i < \omega$, let $\{X_{ij} | j < \omega\}$ be a maximal antichain of $\{Y \subseteq X | Y \Vdash_{\mathcal{S}} t(i) = 0 \text{ or } Y \Vdash_{\mathcal{S}} t(i) = 1\}$. Let $W = \{(i, j) | X_{ij} \Vdash_{\mathcal{S}} t(i) = 1\}$. Take x to be a code for the function $y \mapsto \{i | \exists j [(i, j) \in W \wedge y \in X_{ij}]\}$. Clearly, for r M -random and $r \in X$, $F(x, r, E_r(t))$ holds, where $E_r(t)$ is the evaluation of the term t in $M[r]$. \square

DEFINITION. Let $\{f_n | n < \omega\}$ be a sequence of functions: $2^\omega \rightarrow 2^\omega$, and let $X, Y \subseteq 2^\omega$. Y covers X with respect to $\{f_n\}$ iff $\mu(\{x \in X | \exists n < \omega [f_n(x) \in Y]\}) = \mu(X)$. X is easily covered with respect to $\{f_n\}$ iff $\forall \epsilon > 0 \exists Y [\mu(Y) < \epsilon \wedge Y \text{ covers } X \text{ with respect to } \{f_n\}]$.

The following is purely a theorem of real analysis in ZFC.

LEMMA 3. *Let $\{f_n | n < \omega\}$ be a sequence of measurable functions: $X \rightarrow 2^\omega$, $X \subseteq 2^\omega$, such that $\forall x \in X \forall i, j \in \omega (i \neq j \rightarrow f_i(x) \neq f_j(x))$. Then (a) X is easily covered with respect to $\{f_n\}$. Moreover, (b) if $\epsilon > 0$ there is $X' \subseteq X$ such that $\mu(X') = \mu(X)$ and an open set U with $\mu(U) < \epsilon$ such that for all $x \in X'$, $\{i \in \omega | f_i(x) \in U\}$ is $(n \mapsto n^5/\epsilon)$ -thick.*

REMARK. (a) is not particularly obvious, even for very simple sequences $\{f_n\}$. The reader may wish to consider some examples like $f_n(x) = (1 - 1/(n + 2)) \cdot x$ for $0 < x < 1$. (In this example we are taking “reals” to mean the usual unit interval rather than 2^ω . It is not hard to see that Lemma 3 applies also in this case.)

PROOF OF LEMMA 3. We prove (b) which easily yields (a). It is safe to assume $\epsilon < 1$, or else just take $X' = X$ and $U = 2^\omega$. Also, we can assume $\mu(X) > 0$, otherwise take $X' = U = \emptyset$. For each $n < \omega$ we construct $X_n \subseteq X$ and an open set U_n such that:

- (i) $\mu(X_n) \geq \mu(X) \cdot (1 - 1/(n + 2))^2$,
- (ii) $\mu(U_n) < \epsilon \cdot 1/(n + 2)^2$,
- (iii) $\forall x \in X_n \exists i [(n + 2)^5/\epsilon \leq i < (n + 3)^5/\epsilon \wedge f_i(x) \in U_n]$.

Given such X_n and U_n , as $\sum_{n=0}^\infty (n + 2)^{-2} < \infty$, taking $X' = \{x \in X | x \in X_n \text{ for all large enough } n\}$, $\mu(X') = \mu(X)$. Also, as $\sum_{n=0}^\infty (n + 2)^{-2} < 1$, taking $U = \bigcup_{n < \omega} U_n$, $\mu(U) < \epsilon$. Finally, for $x \in X'$, if m is such that $x \in X_n$ whenever $n > m$, we have that whenever $n > m + 2$,

$$\{i | n^5/\epsilon \leq i < (n + 1)^5/\epsilon\} \cap \{i | f_i(x) \in U\} \neq \emptyset.$$

Fix n . Let $I = \{i | (n + 2)^5/\epsilon \leq i < (n + 3)^5/\epsilon\}$. For $i \in I$, let $\hat{f}_i = \nu \cdot f_i$. \hat{f}_i is a measurable real-valued (in the ordinary sense of real analysis) function on X . Let $A \subseteq X$ be compact with $\mu(A) \geq \mu(X) \cdot (1 - \frac{1}{4} \cdot (n + 2)^{-2})$. By Lusin’s theorem, there is a compact $B \subseteq A$ with $\mu(B) \geq \mu(X) \cdot (1 - \frac{1}{2} \cdot (n + 2)^{-2})$ such that for

each $i \in I, \hat{f}_i$ is continuous on B . Thus the function $x \mapsto \inf\{|\hat{f}_i(x) - \hat{f}_j(x)| \mid i, j \in I, i \neq j\}$ is a continuous, strictly positive function on B . So $\delta \equiv \inf\{|\hat{f}_i(x) - \hat{f}_j(x)| \mid i, j \in I, i \neq j, x \in B\} > 0$. Let Y_0, \dots, Y_{k-1} be a pairwise-disjoint collection of basic open neighborhoods of 2^ω such that each has measure $1/k$, $\cup_{j < k} Y_j = 2^\omega$, and the diameter of $\nu[Y_j] < \delta$. So, for any $x \in B$ and $j < k$ there is at most one $i \in I$ with $f_i(x) \in Y_j$.

Claim. Suppose $W \subseteq B$. Then there is $j < k$ such that

$$\mu(\{x \in W \mid \exists i \in I [f_i(x) \in Y_j]\}) \geq \bar{I} \cdot \mu(W) / k.$$

REMARK. To understand the point of this claim, imagine that $\mu(W) = 1$. Then it says that the amount of W that Y_j “covers” is very large compared to the size of Y_j , namely, \bar{I} times as large. This is the essential combinatorial idea behind Lemma 3.

PROOF OF CLAIM. Let $Z_{ij} = \{x \in W \mid f_i(x) \in Y_j\}$. So for each j , if $i \neq i'$ then $Z_{ij} \cap Z_{i'j} = \emptyset$. Thus

$$\begin{aligned} \mu(\{x \in W \mid \exists i \in I [f_i(x) \in Y_j]\}) &= \sum_{i \in I} \mu(Z_{ij}). \\ \sum_{j < k} \sum_{i \in I} \mu(Z_{ij}) &= \sum_{i \in I} \sum_{j < k} \mu(Z_{ij}) = \bar{I} \cdot \mu(W), \end{aligned}$$

as $\{Z_{ij} \mid j < k\}$ is a partition of W . So for some $j < k$ it must be that $\sum_{i \in I} \mu(Z_{ij}) > \bar{I} \cdot \mu(W) / k$. \square (for claim)

Now, construct a (finite) sequence of subsets of $k, K_0 \subseteq K_1 \subseteq \dots \subseteq K_s$ as follows. Let $K_0 = \emptyset$. Assume K_t has been defined. Let $V_t = \{x \in B \mid \exists i \in I [f_i(x) \in \cup_{j \in K_t} Y_j]\}$. If $\mu(V_t) \geq \mu(X) \cdot (1 - (n + 2)^{-2})$, then let $s = t$ and K_t is the last element of our sequence. Otherwise, let $W = B - V_t$ and let $K_{t+1} = K_t \cup \{j\}$ where j is chosen so that $\mu(\{x \in W \mid \exists i \in I [f_i(x) \in Y_j]\}) \geq \bar{I} \cdot \mu(W) / k$. Obviously $j \notin K_t$. As $\mu(W) \geq \mu(X) \cdot (1/2(n + 2)^2)$,

$$\mu(V_{t+1}) \geq \mu(V_t) + (\bar{I}/k) \cdot \mu(X) \cdot (1/2(n + 2)^2).$$

As k is finite, the iterative procedure eventually yields K_s such that $\mu(V_s) \geq \mu(X) \cdot (1 - (n + 2)^{-2})$.

$$\mu(X) \geq \mu(V_s) \geq s \cdot (\bar{I}/k) \mu(X) \cdot (1/2(n + 2)^2),$$

so letting $U_n = \cup_{j \in K_s} Y_j$, we have $\mu(U_n) = s/k \leq 2(n + 2)^2 / \bar{I}$. As $\bar{I} > 2(n + 2)^4 / \epsilon$, $\mu(U_n) < \epsilon / (n + 2)^2$. Thus we have (ii). Now, taking $X_n = V_s$, we immediately get (i) and (iii). \square (for Lemma 3)

We now prove the Theorem stated in the Abstract. (b) clearly yields (a), so let $\{x_n \mid n < \omega\} \in M[r]$ be a sequence of distinct reals. Let t be a term such that $E_r(t)$ is this sequence, and let X_1 be such that $r \in X_1$ and $X_1 \Vdash_{\mathbb{S}} t$ is a sequence of distinct reals. Let t_n be the term saying “the n th element of t ”. Now, working in M , for each n , let x_n be such that $C(x_n) \wedge X_1 \Vdash_{\mathbb{S}} F(x_n, r, t_n)$. For $x \in X_1$, let $f_n(x) =$ the unique z such that $F(x_n, x, z)$. For $m, n \in \omega, m \neq n$, let $A_{m,n} = \{x \mid f_m(x) = f_n(x)\}$. Suppose $\mu(A_{m,n}) > 0$. Then $M \models \forall x \in A_{m,n} \forall z [F(x_m, x, z) \leftrightarrow F(x_n, x, z)]$. Thus $A_{m,n} \Vdash_{\mathbb{S}} t_m = t_n$, for, if r' is M -random and $r' \in A_{m,n}$, by absoluteness for Π^1_1

sentences, $M[r'] \models \forall x \in A_{m,n} \forall z [F(x_m, x, z) \leftrightarrow F(x_n, x, z)]$ and, as $A_{m,n} \subseteq X_1$, $M[r'] \models F(x_m, r', t_m) \wedge F(x_n, r', t_n)$. This contradicts that $X_1 \Vdash_{\mathbb{S}} t$ is a sequence of distinct reals. Thus $\mu(A_{m,n}) = 0$. Let $X = X_1 - \cup \{A_{m,n} \mid m, n \in \omega, m \neq n\}$. As $\mu(X) = \mu(X_1)$, $r \in X$.

Now, let

$$\mathfrak{D} = \{ W \in \mathfrak{P}_\epsilon \mid \exists X' \subseteq X \exists \epsilon > 0 [\mu(X') = \mu(X) \wedge \forall x \in X' (\{i \mid f_i(x) \in W\} \text{ is } (n \mapsto n^5/\epsilon)\text{-thick})] \}.$$

Lemma 3(b) shows that \mathfrak{D} is dense in \mathfrak{P}_ϵ . As \mathcal{Q} is M -generic on \mathfrak{P}_ϵ , there is $W \in \mathfrak{D}$ with $W \subseteq \mathcal{Q}$. Let X', ϵ be as in the definition of \mathfrak{D} for W . As $\mu(X') = \mu(X)$, $r \in X'$. The statement $\forall x \in X' (\{i \mid f_i(x) \in W\} \text{ is } (n \mapsto n^5/\epsilon)\text{-thick})$ is Π^1_1 , so by absoluteness we get that $M[r] \models \{i \mid x_i \in W\} \text{ is } (n \mapsto n^5/\epsilon)\text{-thick}$. Since for large enough n there is always an m with $2^n < m^5/\epsilon < (m+1)^5/\epsilon < 2^{n+1}$, $M[r] \models \{i \mid x_i \in W\} \text{ is } (n \mapsto 2^n)\text{-thick}$. As $W \subseteq \mathcal{Q}$, we are done. \square (for Theorem)

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