AN IMPROVED ESTIMATE FOR CERTAIN
DIOPHANTINE INEQUALITIES

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ABSTRACT. Let \( \lambda_1, \ldots, \lambda_8 \) be any nonzero real numbers such that not all \( \lambda_j \) are of the same sign and not all ratios \( \lambda_j/\lambda_k \) are rational. If \( \eta, a \) are any real numbers with \( 0 < a < 3/70 \) then \( |\eta + \sum_{j=1}^{8} \lambda_j n_j^3| < (\max n_j)^{-a} \) has infinitely many solutions in positive integers \( n_j \).

1. Introduction. Throughout \( \eta \) is any real number and \( \lambda_1, \ldots, \lambda_8 \) are any nonzero real numbers such that not all \( \lambda_j \) are of the same sign and not all ratios \( \lambda_j/\lambda_k \) are rational. Improving a result of Davenport and Heilbronn [4], Davenport and Roth [5, Theorem 2] proved:

**Theorem DR.** For any \( \epsilon > 0 \) the inequality \( |\eta + \sum_{j=1}^{8} \lambda_j n_j^3| < \epsilon \) has infinitely many solutions in positive integers \( n_j \).

Furthermore, Baker [1] proved that for any positive integer \( N \) the inequality \( |\sum_{j=1}^{8} \lambda_j p_j^3| < (\max \log p_j)^{-N} \) has infinitely many solutions in primes \( p_j \). Results in [4] and [1] were improved and generalized by Danicic [3], Schwarz [9], Ramachandra [8], Vaughan [10], Lau and Liu [6a], [7]. In particular [7, Theorem 2] if

\[ 0 < a < \frac{\sqrt{21} - 1}{15360} (1.1) \]

then the inequality \( |\eta + \sum_{j=1}^{8} \lambda_j p_j^3| < (\max p_j)^{-a} \) has infinitely many solutions in primes \( p_j \). In this paper we shall prove:

**Theorem.** If \( 0 < a < 3/70 \) then

\[ \left| \eta + \sum_{j=1}^{8} \lambda_j n_j^3 \right| < (\max n_j)^{-a} \]  

(1.2)

has infinitely many solutions in positive integers \( n_j \) and no component \( n_j \) is bounded above.

Our Theorem is an improvement of Theorem DR in the error term \( \epsilon \). Also, \( a < 3/70 \) is a more desirable result since it is analogous to (1.1). Furthermore the error term in (1.2) is of the right order of infinity. Indeed we may let \( \eta = 0, \lambda_1 \) be irrational and all other \( \lambda_j \) be integers then (1.2) implies that \( |\lambda_1 + (\sum_{j=2}^{8} \lambda_j n_j^3)/n_1^3| < n_1^{-3-a} \) has infinitely many integer solutions \( n_1^3 \). So in view of Dirichlet's theorem [6, Theorems 193 and 194] we see that the order of infinity of the error term in (1.2) cannot be improved except the bound of \( a \).

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The proof of our theorem follows the basic format of the Davenport-Roth argument [5, §4]; the improvement results from a more careful treatment of the minor arcs (Lemma 9, cf. Lemma 13 of Cook [2]). An alternative method of proving (1.2) with a positive $\alpha$ was outlined by Vaughan [10, p. 177].

Following exactly the same argument as that of the proof of our theorem, we can improve the results in [2] by replacing the $\epsilon$ in Theorems 1 and 2 of [2] by $(\max_{1 < |y| < 6} x_j, y)^{-\beta}$ and $(\max_{1 < |y| < 4} x_j, y_1, y_2)^{-\beta}$ respectively, where $0 < \beta < 1/35$. We shall omit the proof of these results.

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2. Notation and definitions. Let $\epsilon$ be any sufficiently small positive number and $x$ a real variable. Write $e(x) = \exp(i2\pi x)$. By $n$, with or without suffices, we denote positive integers. By the given hypotheses on $\lambda_j$ we may assume (cf. [2, p. 143, §2])

$$\lambda_1/\lambda_2 < 0 \text{ and irrational.}$$  \hspace{1cm} (2.1)

Then by Theorem 183 in [6] there are infinitely many convergents $a/q$ with $1 < q$ and

$$(a, q) = 1, \quad |\lambda_1/\lambda_2 - a/q| < 1/(2q^2).$$  \hspace{1cm} (2.2)

Let $X$ be large so that

$$X = q^{2/3},$$  \hspace{5cm} (2.3)

$$I_j = I_j(x) = \int_{r_jx}^{2r_jx} e(\lambda_j xy^3) \, dy \quad (j = 1, 2),$$  \hspace{1cm} (2.4)

$$S_j = S_j(x) = \begin{cases} \sum_{r_jx < n < 2r_jx} e(\lambda_j xn^3) & (j = 1, 2, 3, 4), \\ \sum_{X^{4/5} < n < 2X^{4/5}} e(\lambda_j xn^3) & (j = 5, 6, 7, 8), \end{cases}$$  \hspace{1cm} (2.5)

where

$$\nu_1 = 1, \quad \nu_2 = |\lambda_1/\lambda_2|^{1/3}, \quad \nu_3 = |\lambda_1/(32\lambda_3)|^{1/3}, \quad \nu_4 = |\lambda_1/(32\lambda_4)|^{1/3}. \hspace{1cm} (2.6)$$

Trivially,

$$|I_j| < \nu_j X \quad (j = 1, 2), \quad |S_j| < \nu_j X \quad (j = 1, 2, 3, 4),$$  \hspace{1cm} (2.7)

$$|S_j| < X^{4/5} \quad (j = 5, 6, 7, 8).$$

Put

$$V(x) = \prod_{j=1}^{8} S_j(x), \quad W(x) = I_1(x)I_2(x) \prod_{j=3}^{8} S_j(x).$$  \hspace{1cm} (2.8)

We dissect the real line into four regions as follows.

$$G_1 = \{ x : |x| < |\lambda_2|^{-1}X^{-2-\epsilon} \}, \quad G_2 = \{ x : |\lambda_2|^{-1}X^{-2-\epsilon} < |x| < X^{3/70} \},$$  \hspace{1cm} (2.9)

$$G_3 = \{ x : X^{3/70} < |x| < X \}, \quad G_4 = \{ x : X < |x| \}.$$
For the given positive $\alpha < 3/70$ let

$$M = 2 \left( \max_{1 < j < 4} \nu_j \right), \quad \tau = (MX)^{-\alpha},$$

$$K_u(x) = \begin{cases} 
    u^2 & \text{if } x = 0, \\
    \left( \sin(\pi ux) / (\pi x) \right)^2 & \text{otherwise},
\end{cases}$$

where $u = \tau$ or 1. Trivially,

$$K_\tau(x) \leq \tau^2.$$  \hspace{1cm} (2.12)

If $U > 0$, we use $V \ll U$ (or $U \gg V$) to denote $|V| < AU$, where $A$ is some positive constant which may depend on $\lambda_j$, $e$ and $\eta$ only.

3. The region $\mathcal{E}_1$.

**Lemma 1.** For any real $y$,

$$\int_{-\infty}^{\infty} e(xy) K_u(x) \, dx = \max(0, u - |y|).$$

**Proof.** It follows from (2.11) and Lemma 4 in [4] by a simple substitution.

**Lemma 2.** For $x \in \mathcal{E}_1$, $S_j(x) = I_j(x) + O(1)$ ($j = 1, 2$).

**Proof.** This is essentially the corollary to Lemma 11 in [5].

**Lemma 3.** If $x \neq 0$ then $I_j(x) \ll X^{-2} |x|^{-1}$ for $j = 1, 2$.

**Proof.** By (2.4) the lemma follows from integration by parts.

**Lemma 4.**

$$\int_{\mathcal{E}_1} V(x)e(x\eta) K_u(x) \, dx = \int_{-\infty}^{\infty} W(x) K_\tau(x) \, dx + O(\tau^2 X^{21/5 - \epsilon}).$$

**Proof.** Note that $e(x\eta) = 1 + O(|x|)$ and $S_1 S_2 - I_1 I_2 = S_1 (S_2 - I_2) + (S_1 - I_1) I_2$. Then by (2.8), Lemma 2, (2.7) and (2.9), for $x \in \mathcal{E}_1$ we have

$$V(x)e(x\eta) - W(x) = (S_1 S_2 - I_1 I_2) \prod_{j=3}^{8} S_j + O(|x| \prod_{j=1}^{8} S_j) \ll X^{31/5}. \hspace{1cm} (3.1)$$

By (3.1), (2.12) and (2.9)$_1$, we see that

$$\int_{\mathcal{E}_1} |V(x)e(x\eta) - W(x)| K_\tau(x) \, dx \ll \tau^2 X^{31/5} \int_{\mathcal{E}_1} dx \ll \tau^2 X^{21/5 - \epsilon}. \hspace{1cm} (3.2)$$

On the other hand, by Lemma 3, (2.8)$_2$, (2.12) and (2.9)$_1$,

$$\int_{x \not\in \mathcal{E}_1} W(x) K_\tau(x) \, dx \ll \tau^2 X^{2 + 16/5} \int_{x \not\in \mathcal{E}_1} (X^2 |x|)^{-2} dx \ll \tau^2 X^{16/5 + \epsilon}. \hspace{1cm} (3.3)$$

Lemma 4 follows from (3.2) and (3.3).

**Lemma 5.** $\int_{\mathcal{E}_1} V(x)e(x\eta) K_\tau(x) \, dx \gg \tau^2 X^{21/5}$. 
Proof. Let
\[ \mathcal{B} = \{ n = (n_3, \ldots, n_8) : \nu_j X < n_j < 2 \nu_j X \ (j = 3, 4), \]
\[ X^{4/5} < n_j < 2X^{4/5} \ (j = 5, 6, 7, 8) \} \tag{3.4} \]
and \( \phi = \lambda_1 y_1 + \lambda_2 y_2 + \sum_{j=3}^8 \lambda_j n_j^3 \) where \( y_j \) are real. It follows from (2.8)2, (2.4), (2.5) and Lemma 1 that
\[ \int_{-\infty}^{\infty} W(x) K_\tau(x) \, dx \]
\[ = \sum_{n \in \mathcal{B}} \int_{\nu_j X}^{8 \nu_j X^3} \int_{X^3}^{8X^3} \left( 3^{-2} (y_1 y_2)^{-2/3} \int_{-\infty}^{\infty} e(x\phi) K_\tau(x) \, dx \right) \, dy_1 dy_2 \]
\[ \gg X^{-4} \sum_{n \in \mathcal{B}} \int_{\nu_j X}^{8 \nu_j X^3} \int_{X^3}^{8X^3} \max(0, \tau - |\phi|) \, dy_1 dy_2. \tag{3.5} \]
If \( 3 \nu_j X^3 < y_j < 6 \nu_j X^3, n \in \mathcal{B} \) and \( |\phi| < \tau/2 = o(1) \), then in view of (2.1), (3.4) and (2.6),
\[ y_1 = |\lambda_2/\lambda_1| y_2 - (\lambda_3/\lambda_1) n_3^3 - (\lambda_4/\lambda_1) n_4^3 - \sum_{j=5}^8 (\lambda_j/\lambda_1) n_j^3 + \phi/\lambda_1 \]
\[ < 6 \nu_j X^3 \lambda_2/\lambda_1 X^3 + |\lambda_3/\lambda_1| 8 \nu_j X^3 + |\lambda_4/\lambda_1| 8 \nu_j X^3 + o(X^3) \]
\[ = 6X^3 + X^3/4 + X^3/4 + o(X^3) < 8X^3. \]
Similarly we have \( y_j > 3X^3 - X^3/4 - X^3/4 + o(X^3) > X^3 \). So by (3.5) and (3.4),
\[ \int_{-\infty}^{\infty} W(x) K_\tau(x) \, dx \gg X^{-4} \sum_{n \in \mathcal{B}} \int_{3 \nu_j X^3}^{6 \nu_j X^3} \int_{-\tau/2}^{\tau/2} (\tau/2) \, d\phi \, dy_2 \gg \tau^2 X^{21/5}. \]
This together with Lemma 4 proves Lemma 5.

4. Some elementary lemmata. For \( j = 1, 2, 3, 4 \) and \( k = 5, 6, 7, 8 \) let
\[ K(g, h) = \int_{-\infty}^{\infty} \left| S_j(x) \right|^g \left| S_k(x) \right|^h K_\tau(x) \, dx, \]
\[ L(g, h) = \int_{-\infty}^{\infty} \left| S_j(x) \right|^g \left| S_k(x) \right|^h K_\tau(x) \, dx. \tag{4.1} \]

Lemma 6. \( K(2, 4) \ll X^{13/5+\varepsilon} \) and \( K(4, 4) \ll X^{21/5+\varepsilon} \).

Proof. These are essentially Lemmata 8 and 10 in [5] respectively.

Lemma 7. \( L(2, 4) \ll \tau X^{13/5+\varepsilon} \) and \( L(4, 4) \ll \tau X^{21/5+\varepsilon} \).

Proof. For the given \( j, k \) implied in \( L(2, 4) \) let
\[ \mathcal{G} = \{ \xi = (n_1, \ldots, n_6) : \nu_j X < n_1, n_2 < 2 \nu_j X, X^{4/5} < n_3, \ldots, n_6 < 2X^{4/5} \} \]
and \( \psi(\xi) = \lambda_j (n_1^3 - n_2^2) + \lambda_k (n_3^3 + n_4^3 - n_3^2 - n_5^2). \) By Lemmata 1, 6 and \( \tau < 1 \), we have
\[ L(2, 4) = \sum_{\xi \in \mathcal{G}} \int_{-\infty}^{\infty} e(x\psi(\xi)) K_\tau(x) \, dx = \sum_{\xi \in \mathcal{G}} \max(0, \tau - |\psi(\xi)|) \]
\[ < \tau \sum_{\xi \in \mathcal{G}} \max(0, 1 - |\psi(\xi)|) = \tau K(2, 4) \ll \tau X^{13/5+\varepsilon}. \]
The inequality for $L(4, 4)$ is proved similarly.

**Lemma 8.** For $j = 1, 2$ let $\lambda_j x = \beta_j + a_j/q_j$, where $a_j, q_j$ are integers with $(a_j, q_j) = 1$. If $\beta_j \ll q_j^{-1}X^{-2-\varepsilon}$, then

(a) $S_j(x) \ll q_j^{-1/3}\min(X, X^{-2}|\beta_j|^{-1})$ when $1 < q_j < X^{1-\varepsilon}$,
(b) $S_j(x) \ll X^{3/4+\varepsilon}$ when $X^{1-\varepsilon} < q_j < X^{2+\varepsilon}$.

**Proof.** Parts (a) and (b) are essentially Lemmata 11 and 12 in [5], respectively.

**Lemma 9.** Let $\rho, \sigma$ be any constants such that $-2 - \varepsilon < \rho < \sigma$ and $0 < \sigma$. If

$|\lambda_2|^{-1}\rho < |x| < X^\sigma$ (4.2)

then $\min(|S_1(x)|, |S_2(x)|) \ll X^{3/4+\varepsilon+\sigma/6}$.

**Proof.** This is a generalization of Lemma 13 in [5]. By Theorem 36 in [6], for each $x$ satisfying (4.2) there are integers $a_j, q_j$ ($j = 1, 2$) with $(a_j, q_j) = 1$ such that

$1 < q_j < X^{2+\varepsilon}$, \hspace{1cm} |q_j\beta_j| < X^{-2-\varepsilon}, \hspace{1cm} (4.3)$

where

$\beta_j = \lambda_j x - a_j/q_j$. \hspace{1cm} (4.4)

We see that $a_2 \neq 0$. For if $a_2 = 0$ then by (4.4) and (4.3), $|\lambda_2 x| = |\beta_2| < X^{-2-\varepsilon}$. This contradicts (4.2).

If $\max(q_1, q_2) > X^{1-\varepsilon}$ then Lemma 9 follows from Lemma 8(b). Suppose that $\max(q_1, q_2) < X^{1-\varepsilon}$. Then Lemma 9 follows from Lemma 8(a) unless the bound of $S_j(x)$ in Lemma 8(a) is $> X^{3/4+\varepsilon+\sigma/6}$ for both $j = 1, 2$. If so then for both $j = 1, 2$ we have

$q_j < X^{3/4-3\varepsilon-\sigma/2}$ and $|\beta_j| < q_j^{-1/3}X^{-11/4-\varepsilon-\sigma/6}. \hspace{1cm} (4.5)$

By (4.4), (4.5) and (2.3),

$|\lambda_1/\lambda_2 a_2 q_1 - a_1 q_2| = q_1 q_2 |\lambda_1/\lambda_2(\lambda_2 x - \beta_2) - (\lambda_1 x - \beta_1)| < q_1 q_2 (|\beta_1| + |\beta_2|) \ll (q_1^{3/2}q_2 + q_2^{3/2}q_1) X^{-11/4-\varepsilon-\sigma/6} \ll X^{-3/2-6\varepsilon-\sigma} < 1/(2q). \hspace{1cm} (4.6)$

Now for any integers $a', q'$ with $1 < q' < q$, it follows from (2.2) that

$|q(\lambda_1/\lambda_2 - a')| > q'\left(\left|\frac{a'q - aq'}{qq'}\right| - \left|\frac{a}{q} - \frac{\lambda_1}{\lambda_2}\right|\right) > q'\left(\frac{1}{qq'} - \frac{1}{2q^2}\right) > \frac{1}{2q}. \hspace{1cm} (4.7)$

Put $q' = |a_2 q_1|$ and $a' = \pm a_1 q_2$. We see that $q' > 1$ as $a_2 \neq 0$. So it follows from (4.6) and (4.7), that

$|a_2 q_1| > q. \hspace{1cm} (4.8)$

On the other hand, by (4.4), (4.5), (4.2) and (2.3),

$|a_2 q_1| = q_1 q_2 |\lambda_2 x - \beta_2| \ll X^{3/2-6\varepsilon-\sigma} X^\sigma < q. \hspace{1cm} (4.9)$

This proves Lemma 9 since (4.8) contradicts (4.9).
5. The regions $G_2$, $G_3$ and $G_4$. Let

$$F_1(x) = |S_1S_2S_6|^2, \quad F_2(x) = |S_2S_3S_6|^2, \quad F_3(x) = |S_3S_4S_7S_8|^2 \quad (5.1)$$

and $\mathcal{R} = \sup_{x \in \mathcal{R}} \min(|S_1(x)|, |S_2(x)|)$ where $\mathcal{R}$ is some region in the real line. By (2.8) and Hölder's inequality we have

$$\int_{\mathcal{R}} |V(x)|K_r(x) \, dx < \mathcal{R} \sum_{m=1}^2 \int_{\mathcal{R}} \prod_{j \neq m} |S_j(x)|K_r(x) \, dx$$

$$< \mathcal{R} \sum_{m=1}^2 \left( \int_{\mathcal{R}} F_m(x)K_r(x) \, dx \right)^{1/2} \left( \int_{\mathcal{R}} F_3(x)K_r(x) \, dx \right)^{1/2} \quad (5.2)$$

**Lemma 10.** $\int_{\mathcal{G}_2} |V(x)|K_r(x) \, dx \ll \tau X^{291/70 + 2\varepsilon}.$

**Proof.** By (5.1), (4.1) and Hölder's inequality we have

$$\int_{\mathcal{G}_2} F_m(x)K_r(x) \, dx \ll L(2, 4) \quad (m = 1, 2) \quad \text{and} \quad \int_{\mathcal{G}_2} F_3(x)K_r(x) \, dx \ll L(4, 4).$$

Then by (5.2), Lemma 9 (with $\rho = -2 - \varepsilon, \sigma = 3/70$) and Lemma 7 we have

$$\int_{\mathcal{G}_2} |V(x)|K_r(x) \, dx \ll X^{3/4 + \varepsilon + 1/140(\tau X^{17/5 + \varepsilon})} \ll \tau X^{291/70 + 2\varepsilon}.$$ 

This proves Lemma 10.

**Lemma 11.** Let $F(x) = \sum e(xf(z_1, \ldots, z_p))$ where $f$ is any real-valued function and the summation is taken over any finite set of values $z_1, \ldots, z_p$. Then for any $B > 4/\tau$,

$$\int_{|x| > B} |F(x)|^2K_r(x) \, dx \ll (\tau B)^{-1} \int_{-\infty}^{\infty} |F(x)|^2K_r(x) \, dx.$$

**Proof.** This is essentially Lemma 2 in [5]. See also Lemma 16 in [7].

**Lemma 12.** $\int_{\mathcal{G}_3} |V(x)|K_r(x) \, dx \ll X^{288/70 + 3\varepsilon}.$

**Proof.** Let $\theta_0 = 3/70$ and $\theta_n = 6\varepsilon + \theta_{n-1}$. Since $\theta_n \to \infty$ as $n \to \infty$ we may let $N$ be the greatest positive integer such that $\theta_{N-1} < 1$. Take $\theta_N = 1$. For each $n < N$ put $\mathcal{G}_n = \{ x: X^\theta_{n-1} < |x| < X^\theta_n \}$. By Lemma 11 (with $B = X^\theta_{n-1}$) and an argument similar to that in Lemma 10 we have for $m = 1, 2$

$$\int_{\mathcal{G}_n} F_m(x)K_r(x) \, dx \ll (\tau X^{\theta_{n-1}})^{-1} \int_{-\infty}^{\infty} F_m(x)K_r(x) \, dx \ll (\tau X^{\theta_{n-1}})^{-1} L(2, 4)$$

as by (2.10) $X^{\theta_{n-1}} > X^{3/70} > 4/\tau$. Similarly we have

$$\int_{\mathcal{G}_n} F_3(x)K_r(x) \, dx \ll (\tau X^{\theta_{n-1}})^{-1} L(4, 4).$$

So by (5.2), Lemma 9 (with $\rho = \theta_{n-1} - \varepsilon, \sigma = \theta_n$) and Lemma 7 we have

$$\int_{\mathcal{G}_n} |V(x)|K_r(x) \, dx \ll X^{3/4 + \varepsilon + \theta_n/6(\tau X^{\theta_{n-1}})^{-1} L(2, 4)^{1/2} L(4, 4)^{1/2}$$

$$\ll X^{3/4 + 2\varepsilon - 5\theta_{n-1}/6\tau^{-1}(\tau X^{17/5 + \varepsilon})} \ll X^{83/20 + 3\varepsilon - 5\theta_0/6} \ll X^{288/70 + 3\varepsilon}.$$
Since $\bigcup_{n=1}^N \mathbb{Q}_n = \mathbb{C}_3$, Lemma 12 follows.

**Lemma 13.** $\int_{\mathbb{Q}_4} |V(x)|K_\varepsilon(x) \, dx \ll X^{16/5+\varepsilon}$.

**Proof.** By (2.8), (2.9), Hölder’s inequality, Lemma 11 (with $B = X$) and Lemma 7 we have

\[
\int_{\mathbb{Q}_4} |V(x)|K_\varepsilon(x) \, dx \\
\ll \left( \int_{|x| > X} \left| S_1S_2S_3S_5 \right|^2K_\varepsilon(x) \, dx \right)^{1/2} \left( \int_{|x| > X} \left| S_4S_5S_7S_8 \right|^2K_\varepsilon(x) \, dx \right)^{1/2} \\
\ll (\tau X)^{-1}L(4, 4) \ll (\tau X)^{-1}X^{21/5+\varepsilon} \ll X^{16/5+\varepsilon}.
\]

This proves Lemma 13.

We come now to prove our theorem. For the given $\alpha$ let $\varepsilon > 0$ satisfy $\alpha + 2\varepsilon < 3/70$. Then it follows from Lemmata 5, 10, 12 and 13 that

\[
\int_{-\infty}^\infty V(x)e(x\eta)K_\varepsilon(x) \, dx \gg \tau^2X^{21/5}.
\]

By Lemma 1, (2.5) and (3.4) this integral is

\[
\sum_{\eta \in \mathbb{Q}} \max_n \left( 0, \tau - \left| \eta + \sum_{j=1}^8 \lambda_j n_j^2 \right| \right) \ll \mathfrak{N},
\]

where $\mathfrak{N}$ is the number of solutions $(n_1, \ldots, n_8)$ of (1.2) with $n_1, \ldots, n_8$ lying in the same range as in the last summation since by (2.10) $\tau < M^{-\alpha}(\max_{1 < j < 8} |n_j|/M)^{-\alpha}$. This completes the proof of our theorem.

**References**


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