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AN IMPROVED ESTIMATE FOR CERTAIN DIOPHANTINE INEQUALITIES

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ABSTRACT. Let $\lambda_1, \ldots, \lambda_8$ be any nonzero real numbers such that not all λ_j are of the same sign and not all ratios λ_j/λ_k are rational. If η, α are any real numbers with $0 < \alpha < 3/70$ then $|\eta + \sum_{j=1}^{8} \lambda_j n_j^3| < (\max n_j)^{-\alpha}$ has infinitely many solutions in positive integers n_j.

1. Introduction. Throughout η is any real number and $\lambda_1, \ldots, \lambda_s$ are any nonzero real numbers such that not all λ_j are of the same sign and not all ratios λ_j/λ_k are rational. Improving a result of Davenport and Heilbronn **[4]**, Davenport and Roth **[5**, Theorem 2] proved:

THEOREM DR. *For any* $\varepsilon > 0$ *the inequality* $|\eta + \sum_{j=1}^{8} \lambda_j n_j^3| < \varepsilon$ *has infinitely many solutions in positive integers* n_j.

Furthermore, Baker **[1]** proved that for any positive integer N the inequality $|\sum_{j=1}^{3} \lambda_j p_j| < (\max \log p_j)^{-N}$ has infinitely many solutions in primes p_j. Results in **[4]** and **[1]** were improved and generalized by Danicic **[3]**, Schwarz **[9]**, Ramachandra **[8]**, Vaughan **[10]**, Lau and Liu **[6a]**, **[7]**. In particular **[7**, Theorem 2] if

$$0 < \alpha < (\sqrt{21} - 1)/15360 \tag{1.1}$$

then the inequality $|\eta + \sum_{j=1}^{9} \lambda_j p_j^3| < (\max p_j)^{-\alpha}$ has infinitely many solutions in primes p_j. In this paper we shall prove:

THEOREM. *If* $0 < \alpha < 3/70$ *then*

$$\left|\eta + \sum_{j=1}^{8} \lambda_j n_j^3\right| < (\max n_j)^{-\alpha} \tag{1.2}$$

has infinitely many solutions in positive integers n_j *and no component* n_j *is bounded above.*

Our Theorem is an improvement of Theorem DR in the error term ε. Also, $\alpha < 3/70$ is a more desirable result since it is analogous to (1.1). Furthermore the error term in (1.2) is of the right order of infinity. Indeed we may let $\eta = 0$, λ_1 be irrational and all other λ_j be integers then (1.2) implies that $|\lambda_1 + (\sum_{j=2}^{8} \lambda_j n_j^3)/n_1^3| < n_1^{-3-\alpha}$ has infinitely many integer solutions n_1^3. So in view of Dirichlet's theorem **[6**, Theorems 193 and 194] we see that the order of infinity of the error term in (1.2) cannot be improved further except the bound of α.

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The proof of our theorem follows the basic format of the Davenport-Roth argument [**5**, §4]; the improvement results from a more careful treatment of the minor arcs (Lemma 9, cf. Lemma 13 of Cook [**2**]). An alternative method of proving (1.2) with a positive α was outlined by Vaughan [**10**, p. 177].

Following exactly the same argument as that of the proof of our theorem, we can improve the results in [**2**] by replacing the ε in Theorems 1 and 2 of [**2**] by $(\max_{1\leqslant j\leqslant 6} x_j, y)^{-\beta}$ and $(\max_{1\leqslant j\leqslant 4} x_j, y_1, y_2)^{-\beta}$ respectively, where $0<\beta<1/35$. We shall omit the proof of these results.

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2. Notation and definitions. Let ε be any sufficiently small positive number and x a real variable. Write $e(x) = \exp(i2\pi x)$. By n, with or without suffices, we denote positive integers. By the given hypotheses on λ_j we may assume (cf. [**2**, p. 143, §2])

$$\lambda_1/\lambda_2 < 0 \text{ and irrational.} \tag{2.1}$$

Then by Theorem 183 in [**6**] there are infinitely many convergents a/q with $1 \leqslant q$ and

$$(a, q) = 1, \qquad |\lambda_1/\lambda_2 - a/q| < 1/(2q^2). \tag{2.2}$$

Let X be large so that

$$X = q^{2/3}, \tag{2.3}$$

$$I_j = I_j(x) = \int_{\nu_j X}^{2\nu_j X} e(\lambda_j x y^3)\, dy \qquad (j = 1, 2), \tag{2.4}$$

$$S_j = S_j(x) = \begin{cases} \displaystyle\sum_{\nu_j X < n \leqslant 2\nu_j X} e(\lambda_j x n^3) & (j = 1, 2, 3, 4), \\ \displaystyle\sum_{X^{4/5} < n \leqslant 2X^{4/5}} e(\lambda_j x n^3) & (j = 5, 6, 7, 8), \end{cases} \tag{2.5}$$

where

$$\nu_1 = 1, \qquad \nu_2 = |\lambda_1/\lambda_2|^{1/3}, \qquad \nu_3 = |\lambda_1/(32\lambda_3)|^{1/3}, \qquad \nu_4 = |\lambda_1/(32\lambda_4)|^{1/3}. \tag{2.6}$$

Trivially,

$$|I_j| \leqslant \nu_j X \quad (j = 1, 2), \qquad |S_j| \leqslant \nu_j X \quad (j = 1, 2, 3, 4),$$
$$|S_j| \leqslant X^{4/5} \quad (j = 5, 6, 7, 8). \tag{2.7}$$

Put

$$V(x) = \prod_{j=1}^{8} S_j(x), \qquad W(x) = I_1(x) I_2(x) \prod_{j=3}^{8} S_j(x). \tag{2.8}$$

We dissect the real line into four regions as follows.

$$\begin{aligned} &\mathfrak{E}_1 = \{x: |x| \leqslant |\lambda_2|^{-1} X^{-2-\varepsilon}\}, \quad \mathfrak{E}_2 = \{x: |\lambda_2|^{-1} X^{-2-\varepsilon} < |x| \leqslant X^{3/70}\}, \\ &\mathfrak{E}_3 = \{x: X^{3/70} < |x| \leqslant X\}, \qquad \mathfrak{E}_4 = \{x: X < |x|\}. \end{aligned} \tag{2.9}$$

For the given positive $\alpha < 3/70$ let

$$M = 2\Big(\max_{1<j<4} \nu_j\Big), \qquad \tau = (MX)^{-\alpha}, \tag{2.10}$$

$$K_u(x) = \begin{cases} u^2 & \text{if } x = 0, \\ (\sin(\pi u x)/(\pi x))^2 & \text{otherwise,} \end{cases} \tag{2.11}$$

where $u = \tau$ or 1. Trivially,

$$K_\tau(x) \leqslant \tau^2. \tag{2.12}$$

If $U > 0$, we use $V \ll U$ (or $U \gg V$) to denote $|V| < AU$, where A is some positive constant which may depend on λ_j, ε and η only.

3. The region $\mathfrak{E}_1$.

Lemma 1. *For any real* y,

$$\int_{-\infty}^{\infty} e(xy)K_u(x)\,dx = \max(0, u - |y|).$$

Proof. It follows from (2.11) and Lemma 4 in [4] by a simple substitution.

Lemma 2. *For* $x \in \mathfrak{E}_1$, $S_j(x) = I_j(x) + O(1)$ $(j = 1, 2)$.

Proof. This is essentially the corollary to Lemma 11 in [5].

Lemma 3. *If* $x \neq 0$ *then* $I_j(x) \ll X^{-2}|x|^{-1}$ *for* $j = 1, 2$.

Proof. By (2.4) the lemma follows from integration by parts.

Lemma 4.

$$\int_{\mathfrak{E}_1} V(x)e(x\eta)K_\tau(x)\,dx = \int_{-\infty}^{\infty} W(x)K_\tau(x)\,dx + O(\tau^2 X^{21/5-\varepsilon}).$$

Proof. Note that $e(x\eta) = 1 + O(|x|)$ and $S_1S_2 - I_1I_2 = S_1(S_2 - I_2) + (S_1 - I_1)I_2$. Then by (2.8), Lemma 2, (2.7) and $(2.9)_1$, for $x \in \mathfrak{E}_1$ we have

$$V(x)e(x\eta) - W(x) = (S_1S_2 - I_1I_2)\prod_{j=3}^{8} S_j + O(|x|)\prod_{j=1}^{8} S_j \ll X^{31/5}. \tag{3.1}$$

By (3.1), (2.12) and $(2.9)_1$, we see that

$$\int_{\mathfrak{E}_1} |V(x)e(x\eta) - W(x)|K_\tau(x)\,dx \ll \tau^2 X^{31/5}\int_{\mathfrak{E}_1} dx \ll \tau^2 X^{21/5-\varepsilon}. \tag{3.2}$$

On the other hand, by Lemma 3, $(2.8)_2$, (2.12) and $(2.9)_1$,

$$\int_{x\notin\mathfrak{E}_1} W(x)K_\tau(x)\,dx \ll \tau^2 X^{2+16/5}\int_{x\notin\mathfrak{E}_1} (X^2|x|)^{-2}\,dx \ll \tau^2 X^{16/5+\varepsilon}. \tag{3.3}$$

Lemma 4 follows from (3.2) and (3.3).

Lemma 5. $\int_{\mathfrak{E}_1} V(x)e(x\eta)K_\tau(x)\,dx \gg \tau^2 X^{21/5}$.

PROOF. Let

$$\mathfrak{B} = \{\mathbf{n} = (n_3, \ldots, n_8): \nu_j X < n_j \leqslant 2\nu_j X \ (j = 3, 4),$$
$$X^{4/5} < n_j \leqslant 2X^{4/5} \ (j = 5, 6, 7, 8)\} \quad (3.4)$$

and $\phi = \lambda_1 y_1 + \lambda_2 y_2 + \sum_{j=3}^{8} \lambda_j n_j^3$ where y_j are real. It follows from $(2.8)_2$, (2.4), (2.5) and Lemma 1 that

$$\int_{-\infty}^{\infty} W(x)K_\tau(x)\,dx$$
$$= \sum_{\mathbf{n}\in\mathfrak{B}} \int_{\nu_2^3X^3}^{8\nu_2^3X^3} \int_{X^3}^{8X^3} \left\{3^{-2}(y_1y_2)^{-2/3} \int_{-\infty}^{\infty} e(x\phi)K_\tau(x)\,dx\right\} dy_1\,dy_2$$
$$\gg X^{-4} \sum_{\mathbf{n}\in\mathfrak{B}} \int_{\nu_2^3X^3}^{8\nu_2^3X^3} \int_{X^3}^{8X^3} \max(0, \tau - |\phi|)\,dy_1\,dy_2. \quad (3.5)$$

If $3\nu_2^3X^3 \leqslant y_2 \leqslant 6\nu_2^3X^3$, $\mathbf{n} \in \mathfrak{B}$ and $|\phi| < \tau/2 = o(1)$, then in view of (2.1), (3.4) and (2.6),

$$y_1 = |\lambda_2/\lambda_1|y_2 - (\lambda_3/\lambda_1)n_3^3 - (\lambda_4/\lambda_1)n_4^3 - \sum_{j=4}^{8} (\lambda_j/\lambda_1)n_j^3 + \phi/\lambda_1$$
$$\leqslant 6\nu_2^3|\lambda_2/\lambda_1|X^3 + |\lambda_3/\lambda_1|8\nu_3^3X^3 + |\lambda_4/\lambda_1|8\nu_4^3X^3 + o(X^3)$$
$$= 6X^3 + X^3/4 + X^3/4 + o(X^3) < 8X^3.$$

Similarly we have $y_1 \geqslant 3X^3 - X^3/4 - X^3/4 + o(X^3) > X^3$. So by (3.5) and (3.4),

$$\int_{-\infty}^{\infty} W(x)K_\tau(x)\,dx \gg X^{-4} \sum_{\mathbf{n}\in\mathfrak{B}} \int_{3\nu_2^3X^3}^{6\nu_2^3X^3} \int_{-\tau/2}^{\tau/2} (\tau/2)\,d\phi\,dy_2 \gg \tau^2X^{21/5}.$$

This together with Lemma 4 proves Lemma 5.

4. Some elementary lemmata. For $j = 1, 2, 3, 4$ and $k = 5, 6, 7, 8$ let

$$K(g, h) = \int_{-\infty}^{\infty} |S_j(x)|^g |S_k(x)|^h K_1(x)\,dx,$$
$$L(g, h) = \int_{-\infty}^{\infty} |S_j(x)|^g |S_k(x)|^h K_\tau(x)\,dx. \quad (4.1)$$

LEMMA 6. *$K(2, 4) \ll X^{13/5+\varepsilon}$ and $K(4, 4) \ll X^{21/5+\varepsilon}$.*

PROOF. These are essentially Lemmata 8 and 10 in [5] respectively.

LEMMA 7. *$L(2, 4) \ll \tau X^{13/5+\varepsilon}$ and $L(4, 4) \ll \tau X^{21/5+\varepsilon}$.*

PROOF. For the given j, k implied in $L(2, 4)$ let

$$\mathfrak{S} = \{\xi = (n_1, \ldots, n_6): \nu_j X < n_1, n_2 \leqslant 2\nu_j X, X^{4/5} < n_3, \ldots, n_6 \leqslant 2X^{4/5}\}$$

and $\psi(\xi) = \lambda_j(n_1^3 - n_2^3) + \lambda_k(n_3^3 + n_4^3 - n_5^3 - n_6^3)$. By Lemmata 1, 6 and $\tau < 1$, we have

$$L(2, 4) = \sum_{\xi\in\mathfrak{S}} \int_{-\infty}^{\infty} e(x\psi(\xi))K_\tau(x)\,dx = \sum_{\xi\in\mathfrak{S}} \max(0, \tau - |\psi(\xi)|)$$
$$\leqslant \tau \sum_{\xi\in\mathfrak{S}} \max(0, 1 - |\psi(\xi)|) = \tau K(2, 4) \ll \tau X^{13/5+\varepsilon}.$$

The inequality for $L(4, 4)$ is proved similarly.

LEMMA 8. *For* $j = 1, 2$ *let* $\lambda_j x = \beta_j + a_j/q_j$, *where* a_j, q_j *are integers with* $(a_j, q_j) = 1$. *If* $\beta_j \ll q_j^{-1} X^{-2-\varepsilon}$, *then*
(a) $S_j(x) \ll q_j^{-1/3} \min(X, X^{-2}|\beta_j|^{-1})$ *when* $1 \leqslant q_j \leqslant X^{1-\varepsilon}$,
(b) $S_j(x) \ll X^{3/4+\varepsilon}$ *when* $X^{1-\varepsilon} < q_j \leqslant X^{2+\varepsilon}$.

PROOF. Parts (a) and (b) are essentially Lemmata 11 and 12 in [5], respectively.

LEMMA 9. *Let* ρ, σ *be any constants such that* $-2-\varepsilon \leqslant \rho < \sigma$ *and* $0 < \sigma$. *If*

$$|\lambda_2|^{-1} X^{\rho} < |x| \leqslant X^{\sigma} \tag{4.2}$$

then $\min(|S_1(x)|, |S_2(x)|) \ll X^{3/4+\varepsilon+\sigma/6}$.

PROOF. This is a generalization of Lemma 13 in [5]. By Theorem 36 in [6], for each x satisfying (4.2) there are integers a_j, q_j ($j = 1, 2$) with $(a_j, q_j) = 1$ such that

$$1 \leqslant q_j \leqslant X^{2+\varepsilon}, \qquad |q_j \beta_j| \leqslant X^{-2-\varepsilon}, \tag{4.3}$$

where

$$\beta_j = \lambda_j x - a_j/q_j. \tag{4.4}$$

We see that $a_2 \neq 0$. For if $a_2 = 0$ then by (4.4) and (4.3), $|\lambda_2 x| = |\beta_2| \leqslant X^{-2-\varepsilon}$. This contradicts (4.2).

If $\max(q_1, q_2) > X^{1-\varepsilon}$ then Lemma 9 follows from Lemma 8(b). Suppose that $\max(q_1, q_2) \leqslant X^{1-\varepsilon}$. Then Lemma 9 follows from Lemma 8(a) unless the bound of $S_j(x)$ in Lemma 8(a) is $> X^{3/4+\varepsilon+\sigma/6}$ for both $j = 1, 2$. If so then for both $j = 1, 2$ we have

$$q_j < X^{3/4-3\varepsilon-\sigma/2} \quad \text{and} \quad |\beta_j| < q_j^{-1/3} X^{-11/4-\varepsilon-\sigma/6}. \tag{4.5}$$

By (4.4), (4.5) and (2.3),

$$\begin{aligned} |(\lambda_1/\lambda_2)a_2 q_1 - a_1 q_2| &= q_1 q_2 |(\lambda_1/\lambda_2)(\lambda_2 x - \beta_2) - (\lambda_1 x - \beta_1)| \ll q_1 q_2 (|\beta_1| + |\beta_2|) \\ &\ll (q_1^{2/3} q_2 + q_2^{2/3} q_1) X^{-11/4-\varepsilon-\sigma/6} \ll X^{-3/2-6\varepsilon-\sigma} < 1/(2q). \end{aligned} \tag{4.6}$$

Now for any integers a', q' with $1 \leqslant q' < q$, it follows from (2.2) that

$$\left| q' \frac{\lambda_1}{\lambda_2} - a' \right| \geqslant q' \left(\frac{|a'q - aq'|}{qq'} - \left| \frac{a}{q} - \frac{\lambda_1}{\lambda_2} \right| \right) > q' \left(\frac{1}{qq'} - \frac{1}{2q^2} \right) > \frac{1}{2q}. \tag{4.7}$$

Put $q' = |a_2 q_1|$ and $a' = \pm a_1 q_2$. We see that $q' \geqslant 1$ as $a_2 \neq 0$. So it follows from (4.6) and (4.7), that

$$|a_2 q_1| \geqslant q. \tag{4.8}$$

On the other hand, by (4.4), (4.5), (4.2) and (2.3),

$$|a_2 q_1| = q_1 q_2 |\lambda_2 x - \beta_2| \ll X^{3/2-6\varepsilon-\sigma} X^{\sigma} < q. \tag{4.9}$$

This proves Lemma 9 since (4.8) contradicts (4.9).

5. The regions $\mathfrak{E}_2$, $\mathfrak{E}_3$ and $\mathfrak{E}_4$. Let

$$F_1(x) = |S_1S_5S_6|^2, \qquad F_2(x) = |S_2S_5S_6|^2, \qquad F_3(x) = |S_3S_4S_7S_8|^2 \tag{5.1}$$

and $\mathfrak{M} = \sup_{x\in\mathfrak{L}} \min(|S_1(x)|, |S_2(x)|)$ where $\mathfrak{L}$ is some region in the real line. By $(2.8)_1$ and Hölder's inequality we have

$$\int_{\mathfrak{L}} |V(x)|K_\tau(x)\,dx < \mathfrak{M} \sum_{m=1}^{2} \int_{\mathfrak{L}} \prod_{j\neq m} |S_j(x)|K_\tau(x)\,dx$$

$$< \mathfrak{M} \sum_{m=1}^{2} \left(\int_{\mathfrak{L}} F_m(x)K_\tau(x)\,dx\right)^{1/2}\left(\int_{\mathfrak{L}} F_3(x)K_\tau(x)\,dx\right)^{1/2}. \tag{5.2}$$

LEMMA 10. $\int_{\mathfrak{E}_2} |V(x)|K_\tau(x)\,dx \ll \tau X^{291/70+2\varepsilon}$.

PROOF. By (5.1), (4.1) and Hölder's inequality we have

$$\int_{\mathfrak{E}_2} F_m(x)K_\tau(x)\,dx \ll L(2,4) \quad (m=1,2) \quad \text{and} \quad \int_{\mathfrak{E}_2} F_3(x)K_\tau(x)\,dx \ll L(4,4).$$

Then by (5.2), Lemma 9 (with $\rho = -2-\varepsilon$, $\sigma = 3/70$) and Lemma 7 we have

$$\int_{\mathfrak{E}_2} |V(x)|K_\tau(x)\,dx \ll X^{3/4+\varepsilon+1/140}(\tau X^{17/5+\varepsilon}) \ll \tau X^{291/70+2\varepsilon}.$$

This proves Lemma 10.

LEMMA 11. *Let $F(x) = \sum e(xf(z_1,\ldots,z_p))$ where f is any real-valued function and the summation is taken over any finite set of values $z_1,\ldots,z_p$. Then for any $B > 4/\tau$,*

$$\int_{|x|>B} |F(x)|^2K_\tau(x)\,dx \ll (\tau B)^{-1}\int_{-\infty}^{\infty} |F(x)|^2K_\tau(x)\,dx.$$

PROOF. This is essentially Lemma 2 in [5]. See also Lemma 16 in [7].

LEMMA 12. $\int_{\mathfrak{E}_3} |V(x)|K_\tau(x)\,dx \ll X^{288/70+3\varepsilon}$.

PROOF. Let $\theta_0 = 3/70$ and $\theta_n = 6\varepsilon + \theta_{n-1}$. Since $\theta_n \to \infty$ as $n\to\infty$ we may let N be the greatest positive integer such that $\theta_{N-1} < 1$. Take $\theta_N = 1$. For each $n \leq N$ put $\mathfrak{L}_n = \{x: X^{\theta_{n-1}} < |x| \leq X^{\theta_n}\}$. By Lemma 11 (with $B = X^{\theta_{n-1}}$) and an argument similar to that in Lemma 10 we have for $m = 1, 2$

$$\int_{\mathfrak{L}_n} F_m(x)K_\tau(x)\,dx \ll (\tau X^{\theta_{n-1}})^{-1}\int_{-\infty}^{\infty} F_m(x)K_\tau(x)\,dx \ll (\tau X^{\theta_{n-1}})^{-1}L(2,4)$$

as by (2.10) $X^{\theta_{n-1}} \geq X^{3/70} > 4/\tau$. Similarly we have

$$\int_{\mathfrak{L}_n} F_3(x)K_\tau(x)\,dx \ll (\tau X^{\theta_{n-1}})^{-1}L(4,4).$$

So by (5.2), Lemma 9 (with $\rho = \theta_{n-1} - \varepsilon$, $\sigma = \theta_n$) and Lemma 7 we have

$$\int_{\mathfrak{L}_n} |V(x)|K_\tau(x)\,dx \ll X^{3/4+\varepsilon+\theta_n/6}(\tau X^{\theta_{n-1}})^{-1}L(2,4)^{1/2}L(4,4)^{1/2}$$

$$\ll X^{3/4+2\varepsilon-5\theta_{n-1}/6}\tau^{-1}(\tau X^{17/5+\varepsilon}) \ll X^{83/20+3\varepsilon-5\theta_0/6}$$

$$\ll X^{288/70+3\varepsilon}.$$

Since $\bigcup_{n=1}^{N} \mathfrak{L}_n = \mathfrak{C}_3$, Lemma 12 follows.

LEMMA 13. $\int_{\mathfrak{C}_4} |V(x)| K_\tau(x)\, dx \ll X^{16/5+\varepsilon}$.

PROOF. By $(2.8)_1$, $(2.9)_4$, Hölder's inequality, Lemma 11 (with $B = X$) and Lemma 7 we have

$$\int_{\mathfrak{C}_4} |V(x)| K_\tau(x)\, dx$$
$$\ll \left(\int_{|x|>X} |S_1 S_2 S_5 S_6|^2 K_\tau(x)\, dx \right)^{1/2} \left(\int_{|x|>X} |S_3 S_4 S_7 S_8|^2 K_\tau(x)\, dx \right)^{1/2}$$
$$\ll (\tau X)^{-1} L(4, 4) \ll (\tau X)^{-1} \tau X^{21/5+\varepsilon} \ll X^{16/5+\varepsilon}.$$

This proves Lemma 13.

We come now to prove our theorem. For the given α let $\varepsilon > 0$ satisfy $\alpha + 2\varepsilon < 3/70$. Then it follows from Lemmata 5, 10, 12 and 13 that

$$\int_{-\infty}^{\infty} V(x) e(x\eta) K_\tau(x)\, dx \gg \tau^2 X^{21/5}.$$

By Lemma 1, (2.5) and (3.4) this integral is

$$\sum_{\substack{\mathbf{n} \in \mathfrak{B} \\ \nu_j X < n_j < 2\nu_j X,\, j=1,2}} \max\left(0, \tau - \left| \eta + \sum_{j=1}^{8} \lambda_j n_j^3 \right| \right) < \tau \mathfrak{N},$$

where $\mathfrak{N}$ is the number of solutions $(n_1, \ldots, n_8)$ of (1.2) with $n_1, \ldots, n_8$ lying in the same range as in the last summation since by (2.10) $\tau < M^{-\alpha}(\max_{1<j<8} n_j/M)^{-\alpha}$. This completes the proof of our theorem.

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