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ALGEBRAIC DEFORMATIONS AND TRIPLE COHOMOLOGY

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ABSTRACT. The fundamental theorems of algebraic deformation theory are shown to hold in the context of enriched triple cohomology. This unifies and generalizes the classical theory.

The fundamental results in algebraic deformation theory connect the low order cohomology groups of an algebra A with the existence of deformations of the algebra structure on A. The theory for associative algebras was initiated by Gerstenhaber in [6], where he also outlined techniques applicable to other special cases (see [7]). Here we give a unified treatment of the deformation theories for a broad class of algebra types using the enrichment over the category of coalgebras and triple cohomology (see [1] and [2]).

Let R be a commutative ring and let (T, μ, η) be a triple on the category **Mod-R**. Recall that a T-algebra structure on an R-module A is given by a map $\alpha: AT \to A$ satisfying the equations:

$$(\alpha)T \cdot \alpha = \mu \cdot \alpha: AT^2 \to A, \qquad \eta \cdot \alpha = \mathrm{id}: A \to A. \tag{1}$$

Following Gerstenhaber, we would like to investigate when a formal power series $\alpha + \Sigma \alpha_n x^n$ with coefficients in $\mathrm{Hom}_R(AT, A)$ determines a formal T-algebra structure on A (or a T-algebra structure on $A[[x]]$, see [6]). Direct use of the conditions (1) immediately brings up the problem of the nonadditivity of most triples of interest, e.g. the tensor-algebra triple. To circumvent this problem we must use an additive enrichment of T over the category of R-coalgebras. We shall assume that the algebras for T are definable by a set of finitary multilinear operations (see [1] or [5]) which is the case for most categories of interest, e.g. associative algebras, commutative algebras, Lie algebras, etc.

Coalgebras and enrichments. Recall that an R-coalgebra (C, δ, ε) is an R-module C equipped with maps $\delta: C \to C \otimes C$ and $\varepsilon: C \to R$ satisfying the equations $\delta \cdot (\mathrm{id} \otimes \delta) \simeq \delta \cdot (\delta \otimes \mathrm{id})$, $\delta \cdot (\mathrm{id} \otimes \varepsilon) \simeq \mathrm{id} \simeq \delta \cdot (\varepsilon \otimes \mathrm{id})$, and $\delta \cdot \tau = \delta$ where τ is the "twist" isomorphism $C \otimes C \to C \otimes C$, and the other isomorphisms are the obvious canonical ones. The category of all R-coalgebras and their structure preserving maps will be denoted **Coalg.** This is cartesian closed and serves as a base category over which most categories of R-algebras are enriched (see [1] or [5]).

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Note that the "cofree-coalgebra" functor is the right adjoint to the obvious forgetful functor from **Coalg** to **Mod-R**.

A *point* in a coalgebra C is an element f satisfying $f\delta = f \otimes f$ and $f\varepsilon = 1$ in R. An N-*deformation* of a point f_0 is a sequence of elements (f_n), $0 < n \leqslant N$, in C satisfying the equations:

$$f_n\delta = \sum_{i+j=n} f_i \otimes f_j, \qquad f_n\varepsilon = 0 \quad (\text{if } n > 0). \tag{2}$$

An ∞-deformation of f_0 determines a family of *formal points* in C; for any r in R, the sum $F = \Sigma_0^\infty f_n r^n$ satisfies $F\delta = F \otimes F$ and $F\varepsilon = 1$. Given some sense of convergence in C, as in the analytic theory, these may determine actual points in C (see **[6]**).

If A and B are R-modules, we denote by (A, B) the cofree-coalgebra over $\mathrm{Hom}_R(A, B)$. The adjunction $\hat{}$: $(A, B) \to \mathrm{Hom}_R(A, B)$ induces an evaluation map $A \otimes (A, B) \to B$, and we may view $(-, -)$ as a generalized Hom functor defining an enrichment of **Mod-R** over **Coalg**. Given a map f: $A \to B$, we say an element g of (A, B) *represents* f if $\hat{g} = f$. There is unique point representing each map $A \to B$, but also for each d: $A \to B$ and each point f in (A, B) there is a 1-deformation of f representing d.

If A_α and B_β are T-algebras (i.e. α: $AT \to A$ and β: $BT \to B$), there is a subcoalgebra of (A, B) whose points precisely represent the T-algebra maps from A_α to B_β (see **[4]**). We denote this coalgebra by (A_α, B_β); this is the **Coalg**-valued Hom from **T-alg.** If d is a 1-deformation of f in (A_α, B_β), $\hat{d}$ is a derivation from A_α to B_β along the algebra map $\hat{f}$.

Though generally there is no linear map $\mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(AT, BT)$ defining T, there is always a natural enrichment of T over **Coalg**, $\mathfrak{T}$: $(A, B) \to (AT, BT)$. If f is a point in (A, B), then $\hat{f}T = \widehat{f\mathfrak{T}}$. If d is a 1-deformation of f, $d\mathfrak{T}$ represents the unique $\hat{f}T$-derivation from AT to BT (viewed as algebras) that coincides with d on A.

If A, B, and C are R-modules, the composition map in **Coalg** $\circ$: $(A, B) \otimes (B, C) \to (A, C)$ induces a composition of deformations defined by convolution, i.e. if $f = (f_n)$ is a deformation in (A, B) and $g = (g_n)$ is a deformation in (B, C), we define $(f \cdot g)_n$ to be $\Sigma_{i+j=n} f_i \circ g_j$ in (A, C).

The deformations. We define the coalgebra S to be the equalizer of the following pair of maps in **Coalg**:

$$(AT, A) \underset{\mu\cdot(\ )}{\overset{(\ )\mathfrak{T}\cdot(\ )}{\rightrightarrows}} (AT^2, A) \tag{3}$$

where $(\ )\mathfrak{T} \cdot (\ ) = \delta \cdot (\mathfrak{T} \otimes \mathrm{id}) \cdot \circ$. The points in S precisely represent the T-algebra structures on A, since T and $\mathfrak{T}$ coincide on points. Thus an ∞-deformation (α_n) of α in S is a deformation of α towards another T-algebra structure on A; the series $\alpha + \Sigma\hat{\alpha}_n x^n$ is a deformation of α in the sense of Gerstenhaber, as outlined at the beginning of this paper.

Let α be a point in S. Two ∞-deformations $\alpha_* = (\alpha_n)$ and $\alpha'_* = (\alpha'_n)$ of α in S are said to be *equivalent* if there exists an ∞-deformation a_* of the identity point in

(A, A) such that $\alpha_* \cdot a_* = a_*\mathfrak{T} \cdot \alpha'_*$. In this case, a_* is a T-algebra isomorphism of the formal T-algebra structures defined by α_* and α'_* on A (the ∞-deformations of the identity forming a group under convolution). A deformation is *trivial* if it is equivalent to the deformation $(\alpha, 0, 0, \dots)$.

Now consider an ∞-deformation (α_n) in S. From (3) we have $\mu \cdot \alpha_n = \Sigma_{i+j=n}\alpha_i\mathfrak{T} \cdot \alpha_j$ for each α_n, i.e.

$$\alpha_n\mathfrak{T} \cdot \alpha - \mu \cdot \alpha_n + \alpha\mathfrak{T} \cdot \alpha_n = -\sum_{\substack{i+j=n \\ i\neq 0\neq j}} \alpha_i\mathfrak{T} \cdot \alpha_j \tag{4}$$

The left side of the above is a coboundary, but to understand in what sense it is one, we must consider

The enriched cohomology complex. If G is the cotriple on **T-Alg** associated with T, the T-algebra A_α generates a resolution of T-algebras $A_\alpha G^* \to A_\alpha$ (see [2]). Hom-ing into a T-algebra B_β yields a complex of coalgebras $(A_\alpha, B_\beta) \to (A_\alpha G^*, B_\beta)$ which we may resolve in **Mod-R.** This defines the *enriched cohomology* groups of A_α with coefficients in B_β. Given a T-algebra map λ: $A_\alpha \to B_\beta$ and looking at the cohomology of the complex restricted to 1-deformations over λ yields the usual triple cohomology groups, as defined by Barr and Beck in [2] (since λ gives B_β the structure of an A_α-module and the 1-deformations correspond to R-linear derivations).

The complex $(A_\alpha G^*, B_\beta)$ is isomorphic (through adjointness, see [3]) to a "nonhomogeneous" complex of coalgebras, yielding the sequence of R-modules and boundary maps as follows:

$$(A, B) \xrightarrow{\partial^0} (AT, B) \xrightarrow{\partial^1} (AT^2, B) \xrightarrow{\partial^2} (AT^3, B) \dots, \tag{5}$$

$$(\ )\partial^n = \alpha\mathfrak{T}^n \cdot (\ ) - \sum_{i=0}^{n-1} (-1)^i\mathfrak{T}^i\mu\mathfrak{T}^{n-1-i} \cdot (\ ) - (-1)^n(\ )\mathfrak{T} \cdot \beta. \tag{6}$$

Again, restricting to 1-deformations over $\lambda U = \xi \in (A, B)$ yields the Barr-Beck (B-B) cohomology groups of A_α with coefficients in B_β, here denoted $H^n_\xi(\alpha, \beta)$. Note that in the following paragraphs the words "cochain", "cocycle", and "coboundary" refer to the Barr-Beck concepts unless otherwise stated. If $\alpha = \beta$ and $\xi = \mathrm{id}$, then the cohomology groups are denoted $H^n(\alpha, \alpha)$.

The connection. Reexamining (2) and (4), we see that the first nonzero term in the deformation $\alpha_* = (\alpha_n)$ is a 1-cocycle, an element of $Z^1(\alpha, \alpha)$.

1. PROPOSITION. *Every ∞-deformation is equivalent to a deformation whose first nonzero term is not a coboundary.*

PROOF. Let α_k be the term in question. If α_k is a coboundary, let x be the 0-cochain such that $x\partial^0 = \alpha_k$. Define an ∞-deformation $a_* = (a_n)$ of the identity by: $a_k = x$ and $a_n = 0$ for $n \neq k$. Letting $\alpha'_* = a_*\mathfrak{T} \cdot \alpha_* \cdot a_*^{-1}$, we find that $\alpha'_n = 0$ for $n < k$, and $\alpha'_k = \Sigma_{h+i+j=k}a_h\mathfrak{T} \cdot \alpha_i \cdot a_j^{-1} = \alpha_k - x\partial^0 = 0$. α'_* is equivalent to α_* by construction; the result follows by induction.

2. COROLLARY. *If $H^1(\alpha, \alpha) = 0$ every deformation of α is trivial.*

We now ask when a 1-deformation may be extended to an ∞-deformation. More generally, suppose $(\alpha_i)_{i<n}$ is an $(n-1)$-deformation of α. If α_n (extending the sequence in S) is to exist it must satisfy condition (4), i.e.

$$\alpha_n \partial^1 = - \sum_{\substack{i+j=n \\ n \neq 0 \neq j}} \alpha_i \mathfrak{T} \cdot \alpha_j.$$

This sum is the *obstruction* in (AT^2, A) to extending $(\alpha_i)_{i<n}$.

3. Lemma. *The obstruction to extending a truncated deformation is a cocycle in the enriched cohomology.*

Proof. A trivial computation of ∂^2 acting on the obstruction, using (4) and the naturality of μ.

Let $\text{obs}(\alpha_n)$ denote the obstruction defined above, and let x_n be any element of (AT, A) such that

$$x_n \delta = x_n \otimes \alpha_0 + \alpha_0 \otimes x_n + \sum_{\substack{i+j=n \\ i \neq 0 \neq j}} \alpha_i \otimes \alpha_j$$

(there are many of these because (AT, A) is cofree). Then $x_n \partial^1 - \text{obs}(\alpha_n)$ is a 2-cochain, and in fact is a cocycle. The class of $x_n \partial^1 - \text{obs}(\alpha_n)$ (which does not depend on the choice of x_n) is the B-B obstruction. If y is a 1-cochain such that $y\partial^1 = x_n \partial^1 - \text{obs}(\alpha_n)$, then defining $\alpha_n = x_n + y$ extends $(\alpha_i)_{i<n}$, so we have

4. Proposition. *A truncated deformation may be extended if and only if its B-B obstruction vanishes.*

5. Corollary. *If $H^2(\alpha, \alpha) = 0$ then every* 1-*cocycle extends to an* ∞-*deformation of* α.

Cohomology of the deformed algebra. Consider deformations α_* of α and β_* of β, and a formal map $\xi_* = (\xi_n)$ from α_* to β_*, i.e. ξ_* is an ∞-deformation of ξ_0 in (A, B) and $\alpha_* \cdot \xi_* = \xi_* \mathfrak{T} \cdot \beta_*$. Noting that $\alpha \cdot \xi_0 = \xi_0 \mathfrak{T} \cdot \beta$ we may now compare $H^n_{\xi_*}(\alpha_*, \beta_*)$ with $H^n_\xi(\alpha, \beta)$; the classical results here (Gerstenhaber [8], Coffee [4]) state that $\dim H^n(\alpha_*, \alpha_*) < \dim H^n(\alpha, \alpha)$, where these are vector spaces over appropriate fields.

Now $H^n_{\xi_*}(\alpha_*, \beta_*)$ is defined by a complex such as (6) where α is replaced by α_*, β by β_*, and composition involves convolution of sequences. The boundary operator here will be denoted ∂_*, the cocycles C_*, etc.

Definition. An *N-cochain* (over ξ_*) is a sequence $(\xi_{1n}) = \xi_{1*}$ of elements in (AT^N, B) such that

$$\xi_{1n} \delta = \sum_{i+j=n} \xi_{1i} \otimes \xi_j^N + \xi_j^N \otimes \xi_{1i}, \qquad \xi_{1n} \varepsilon = 0. \tag{7}$$

(Here ξ_*^N equals ξ_* in dimension 0, $\alpha_* \cdot \xi_*$ in dimension 1, etc. We usually omit this superscript N.) ξ_{1*} is an *N-cocycle* if $\xi_{1*} \partial_*^N = 0$, i.e.

$$\sum_{i+j=n} \alpha_i \mathfrak{T}^N \cdot \xi_{1j} - (-1)^N \xi_{1j} \mathfrak{T} \cdot \beta_i - \sum_{i=0}^{N-1} (-1)^i \mathfrak{T}^i \mu \mathfrak{T}^{n-1-i} \cdot (\xi_{1n}) = 0. \tag{8}$$

If $\xi_{1*} \in Z_*$, in particular we have $\xi_{10}\delta = \xi_{10} \otimes \xi_0 + \xi_0 \otimes \xi_{10}$ and $\xi_{10}\partial = 0$, i.e. ξ_{10} is a cocycle over ξ_0. Thus $\xi_{1*} \mapsto \xi_{10}$ defines a map $Z_* \to Z$, the image of which is the module of *liftable* cocycles, denoted Z_l. Obviously, this map carries coboundaries to coboundaries, so we have an induced map $H_* \to H$, the image of which is the module of liftable classes, denoted H_l.

Letting $Z_n = \{\xi_{1*} \in Z_*\colon \xi_{1i} = 0,\ i < n\}$ defines a filtering $Z_* = Z_0 \supset Z_1 \supset Z_2 \ldots$. Notice that if $\xi_{1*} \in Z_*$, then so is $\xi_{1*}^{(k)}$ defined by

$$\xi_{1n}^{(k)} = \begin{cases} 0, & n < k, \\ \xi_{1(n-k)}, & n \geqslant k; \end{cases}$$

this defines an isomorphism $Z_0 \tilde{\to} Z_k$. Now it is obvious that $Z_l \simeq Z_0/Z_1$, and in fact $Z_l \simeq Z_n/Z_{n+1}$ for every $n \geqslant 0$.

6. **Proposition.** *Let $H_* = H_0 \supset H_1 \supset H_2 \ldots$ be the induced filtering of H_*. Then there are epimorphisms $H_l \to H_n/H_{n+1}$ for all $n \geqslant 0$.*

Proof. Again we have $H_l \simeq H_0/H_1$, and the map $H_0 \to H_k$ induced by $Z_0 \to Z_k$ is an epimorphism.

Note that the epimorphisms mentioned above may not be monomorphisms. Pertinent to this is the question of when an n-truncated N-cocycle $(\xi_{1i})_{i<n}$ may be extended to an element of Z_*^N. If the continuation, ξ_{1n}, is to exist it must satisfy (7) and (8), from which we get

$$\sum_{\substack{i+j=n\\ i\neq 0}} -\alpha_i \mathfrak{T}^N \cdot \xi_{1j} + (-1)^N \xi_{1j} \mathfrak{T} \cdot \beta_i = \xi_{1n}\partial^N.$$

The sum on the left of the above equation is the obstruction, $\mathrm{obs}(\xi_{1n})$, in (AT^{N+1}, B) to extending $(\xi_{1i})_{i<n}$. It is easy to show that the obstruction is a cocycle in the enriched cohomology. As in the case of a truncated deformation, it defines a class in H^{N+1} which must vanish if ξ_{1n} is to exist. More precisely, let x_{1n} be any element of (AT^N, B) such that

$$x_{1n}\delta = x_{1n} \otimes \xi_0 + \xi_0 \otimes x_{1n} + \sum_{\substack{i+j=n\\ i\neq 0}} \xi_{1i} \otimes \xi_j + \xi_j \otimes \xi_{1i}.$$

Then the class of $x_{1n}\partial - \mathrm{obs}(\xi_{1n})$ in H^{N+1} is the B-B obstruction.

7. **Proposition.** *If an n-truncated cocycle is obstructed, then the epimorphism $H_l \to H_n/H_{n+1}$ has a nontrivial kernel.*

Proof. Consider any element $\lambda_{1*} \in C_*$ such that

$$\lambda_{1i} = \begin{cases} \xi_{1i}, & i < n, \\ x_{1n}, & i = n, \end{cases}$$

and look at $\lambda_{1*}\partial_*$; it is easy to see that its first $n - 1$ terms are 0 while its nth term is $x_{1n}\partial - \mathrm{obs}(\xi_{1n})$. Thus $x_{1n}\partial - \mathrm{obs}(\xi_{1n}) \in H_l$ but its image in H_n/H_{n+1} is the class of $\lambda_{1*}\partial_*$.

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