PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 4, April 1980

ON SEMIHEREDITARY NONCOMMUTATIVE POLYNOMIAL RINGS

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ABSTRACT. McCarthy [4] showed that a polynomial ring over a commutative von Neumann regular ring is semihereditary. Camillo [1] proved the converse. In this paper we examine polynomial rings over von Neumann regular rings which are not necessarily commutative.

By a ring R we shall mean an associative ring with unit element. R is von Neumann regular if for each $a \in R$, there is an a' in R such that $aa'a = a$. Then $e = aa'$ is an idempotent and $aR = eR$. A ring S is right (resp. left) semihereditary if each of its finitely generately right (resp. left) ideals is projective as an S-module. Let $S = R[x]$ be the polynomial ring in the (commuting) indeterminate x.

THEOREM. *The following are equivalent.*

(a) *R is von Neumann regular.*

(b) *For each $a \in R$, $aS + xS$ is a projective right ideal of S.*

(c) *For each $a \in R$, $Sa + Sx$ is a projective left ideal of S.*

PROOF. Let R be von Neumann regular. If $a \in R$ then there exists an $a' \in R$ satisfying $aa'a = a$. Let $e = aa'$. From the equations

$$a = (e + (1 - e)x)a, \qquad x = (e + (1 - e)x)(1 - e + ex),$$
$$e + (1 - e)x = aa' + x(1 - e),$$

we deduce that $aS + xS = (e + (1 - e)x)S$. It is easily verified that $f = e + (1 - e)x$ is a regular element (= nonzero-divisor) of S so that left multiplication by f induces an isomorphism between S and fS. Hence $aS + xS$ is projective, proving (a) $\Rightarrow$ (b).

Suppose that for all $a \in R$, $aS + xS$ is projective. Fix $a \in R$ and let $K = aS + xS$. By the dual basis lemma for projective modules (see for example [2, p. 141]) there exist S-homomorphisms α and β from K into S, such that for every $k \in K$, $k = a\alpha(k) + x\beta(k)$. In particular, $a - a\alpha(a) = x\beta(a)$. Since x is central in S and α is an S-homomorphism, $x\alpha(a) = \alpha(a)x = \alpha(x)a$ so that $ax - a\alpha(x)a = x^2\beta(a)$. Equating coefficients of x on both sides, we obtain $a = aa'a$ where a' is the coefficient of x of the polynomial $\alpha(x)$. Hence R is von Neumann regular, proving (b) $\Rightarrow$ (a). The equivalence of (a) and (c) now follows from the left-right symmetry of (a).

Received by the editors February 26, 1979 and, in revised form, May 25, 1979.
AMS (MOS) subject classifications (1970). Primary 13B25; Secondary 16A30.
Key words and phrases. Polynomial ring, noncommutative, von Neumann regular.
[1]Research supported by grants from C.S.I.R. and Anglo-American Corporation (South Africa).

COROLLARY. *If $R[x]$ is either left or right semihereditary, then R is a von Neumann regular ring.*

We remark that the corollary follows immediately from a result of Jensen [3], where it is proved that for any ring R w.gl.dim $R[x]$ = w.gl.dim $R + 1$.

If $R[x]$ is semihereditary, then w.gl.dim $R[x] \leq 1$ so that w.gl.dim $R = 0$, from which it follows that R is von Neumann regular.

ACKNOWLEDGEMENT. The author is deeply indebted to Professor Carl Faith for many stimulating discussions during the period in which this paper was written.

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