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# FINITELY GENERATED RIGHT IDEALS OF TRANSFORMATION NEAR-RINGS

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ABSTRACT. If $V$ is an additive group, $M_0(V)$ the near-ring of zero-fixing maps of $V$ into $V$, then the finitely generated right ideals of $M_0(V)$ are easily characterised. These are just the annihilators of subsets of $V$. Moreover, finitely generated right ideals of $M_0(V)$ are generated by a single element.

Throughout this paper all groups will be written additively but this does not imply commutativity. All near-rings considered will be zero-symmetric and left distributive. The near-ring of all zero-fixing maps of a group $V$ into itself under composition and pointwise addition will be denoted by $M_0(V)$.

Let $N$ be a near-ring, and $S$ a nonempty subset of $N$. The right ideal of $N$ generated by $S$ will be denoted by $R(S)$. If $H$ is a right ideal of $N$ and $H = R(S)$ where $S$ is a finite nonempty subset of $N$, then the right ideal $H$ will be said to be *finitely generated*. In the case where $S$ consists of the single element $\gamma$ of $N$, $R(S)$ will be denoted by $R(\gamma)$. Clearly if $S = \{\gamma_1, \dots, \gamma_k\}$ where $k > 1$ is an integer, then $R(S) = R(\gamma_1) + R(\gamma_2) + \cdots + R(\gamma_k)$. The following theorem will be proved.

THEOREM 1. *Let $V$ be a group and $H$ a right ideal of $M_0(V)$. The right ideal $H$ is finitely generated if, and only if, $H = (0 : L)$ where $L$ is some nonempty subset of $V$. Furthermore if $H$ is finitely generated then there exists $\gamma$ in $H$ such that $H = R(\gamma)$.*

This theorem will be proved with the aid of certain propositions and lemmas.

Let $V$ be a group. Suppose $S_1$ and $S_2$ are subsets of $V$ such that $S_1 \cup S_2 = V$ and $S_1 \cap S_2 = \{0\}$ then, as every function $\alpha$ of $V$ into $V$ that fixes zero can be expressed as a sum $\alpha_1 + \alpha_2$, where $\alpha_i$, $i = 1, 2$, are functions of $V$ into $V$ fixing zero and $\alpha_1$ is zero on $S_1$ and $\alpha_2$ is zero on $S_2$, the following proposition holds.

PROPOSITION 2. *Let $V$ be a group and $S_1$ and $S_2$ subsets of $V$ such that $S_1 \cup S_2 = V$, and $S_1 \cap S_2 = \{0\}$. It follows that $M_0(V) = (0 : S_1) \oplus (0 : S_2)$.*

Suppose $L$ is a nonempty subset of a group $V$. Let $L_1 = L \cup \{0\}$, and $L_2 = (V \setminus L) \cup \{0\}$. Clearly the subsets $L_1$ and $L_2$ of $V$ satisfy the conditions of the above proposition and therefore, $(0 : L_1) \oplus (0 : L_2) = M_0(V)$. The identity 1 of $M_0(V)$ can therefore be written as $e_1 + e_2$ where $e_1$ is in $(0 : L_1)$, and $e_2$ is in $(0 : L_2)$. Also if $\alpha$ is in $M_0(V)$, then $e_1\alpha + e_2\alpha = \alpha$. Thus $e_1M_0(V) = (0 : L_1)$, and $(0 : L_1)$ $(= (0 : L))$ is generated by the single element $e_1$ (if $L = \{0\}$, or $V$, then

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$(0 : L)$ is $M_0(V)$, or $\{0\}$, and the result holds with $e_1 = 1$, or 0). It now follows that right ideals of $M_0(V)$ of the form $(0 : L)$ ($L$ a nonempty subset of $V$) are finitely generated and the 'if' part of Theorem 1 is established.

If $\gamma$ is an element of $M_0(V)$ then we denote the set $\{v \in V : v\gamma = 0\}$ by $Z(\gamma)$ (see [**2**, p. 200]).

LEMMA 3. *If $V$ is a group and $\gamma$ an element of $M_0(V)$ then $R(\gamma) = (0 : Z(\gamma))$.*

PROOF. Let $\Lambda = (V \setminus Z(\gamma)) \cup \{0\}$. By Proposition 2, $M_0(V) = (0 : Z(\gamma)) \oplus (0 : \Lambda)$. Clearly $\gamma$ is in $(0 : Z(\gamma))$ and, $R(\gamma) \leqslant (0 : Z(\gamma))$. Now $1 = e_1 + e_2$, where $e_1$ is in $(0 : Z(\gamma))$, and $e_2$ is in $(0 : \Lambda)$. Also, since, $e_2 M_0(V) = (0 : \Lambda)$, $R(e_2) = (0 : \Lambda)$. Consider the element $\gamma + e_2$ of $M_0(V)$. If $v$ is in $(V \setminus Z(\gamma)) \cup \{0\}$, then $ve_2 = 0$ and $v\gamma \neq 0$, unless $v = 0$. If $v$ is in $Z(\gamma)$, then since $Z(e_2) = (V \setminus Z(\gamma)) \cup \{0\}$, it follows that $ve_2 \neq 0$, unless $v = 0$. Thus if $v$ in $V$ is such that $v(\gamma + e_2) = 0$, then it must follow that $v = 0$. Now $R(\gamma + e_2) = M_0(V)$, or $R(\gamma + e_2)$ is contained in a maximal right ideal of $M_0(V)$. If $R(\gamma + e_2)$ is contained in a maximal right ideal of $M_0(V)$ by a result of M. Johnson [**1**] (see also Pilz [**2**, p. 200]) it follows that there exists $v \neq 0$ in $V$ such that $v(\gamma + e_2) = 0$. Thus $R(\gamma + e_2) = M_0(V)$. Now $R(\gamma + e_2) \leqslant R(\gamma) + R(e_2)$, and since $R(\gamma) \leqslant (0 : Z(\gamma))$ and $R(e_2) = (0 : \Lambda)$, we conclude that $R(\gamma) \oplus (0 : \Lambda) = M_0(V)$. Clearly this can only happen if $R(\gamma) = (0 : Z(\gamma))$. The lemma is therefore proved.

COROLLARY. *If $\lambda_1$ and $\lambda_2$ are elements of $M_0(V)$ such that $Z(\lambda_1) < Z(\lambda_2)$, then $R(\lambda_2) < R(\lambda_1)$.*

By the above lemma, if the last statement of the theorem is proved the rest will follow.

LEMMA 4. *Let $V$ be a group and $H$ a finitely generated right ideal of $M_0(V)$. There exists $\gamma$ in $H$ such that $R(\gamma) = H$.*

PROOF. Suppose $P$ is a right ideal of $M_0(V)$ generated by a two element subset $\{\gamma_1, \gamma_2\}$ of $M_0(V)$. If it is established that $P = R(\gamma')$ for some $\gamma'$ in $M_0(V)$, then the lemma will follow. Let $\Lambda = (V \setminus Z(\gamma_1)) \cup \{0\}$. We have, by Proposition 2, that $M_0(V) = (0 : Z(\gamma_1)) \oplus (0 : \Lambda)$, and by Lemma 3, $R(\gamma_1) = (0 : Z(\gamma_1))$. Now $\gamma_2 = \alpha + \beta$ where $\alpha$ is in $(0 : Z(\gamma_1))$, and $\beta$ is in $(0 : \Lambda)$. Take $\gamma' = \gamma_1 + \beta$. Now every $v$ in $V$ is such that either $v\gamma_1 = 0$ or $v\beta = 0$. Thus if $v\gamma' = 0$, it follows that $v\gamma_1 + v\beta = 0$, and $v\gamma_1 = 0$. Hence $Z(\gamma') \subseteq Z(\gamma_1)$, and by the Corollary of Lemma 3, we conclude that $R(\gamma_1) \leqslant R(\gamma')$. Thus $-\gamma_1 + \gamma' = \beta$ is in $R(\gamma')$, and since $\alpha$ is in $(0 : Z(\gamma_1)) = R(\gamma_1)$, we see that $\alpha + \beta = \gamma_2$ is in $R(\gamma')$. Hence $R(\gamma') > R(\gamma_1) + R(\gamma_2) \geqslant P$. But $R(\gamma') \leqslant R(\gamma_1) + R(\beta)$ and, since $\alpha$ is in $R(\gamma_1)$, $R(\gamma') < R(\gamma_1) + R(\gamma_2)$ $(= P)$. Thus $R(\gamma') = P$ and the lemma follows.

## REFERENCES

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