FINITELY GENERATED RIGHT IDEALS OF
TRANSFORMATION NEAR-RINGS

S. D. SCOTT

ABSTRACT. If V is an additive group, $M_0(V)$ the near-ring of zero-fixing maps of V into V, then the finitely generated right ideals of $M_0(V)$ are easily characterised. These are just the annihilators of subsets of V. Moreover, finitely generated right ideals of $M_0(V)$ are generated by a single element.

Throughout this paper all groups will be written additively but this does not imply commutativity. All near-rings considered will be zero-symmetric and left distributive. The near-ring of all zero-fixing maps of a group V into itself under composition and pointwise addition will be denoted by $M_0(V)$.

Let N be a near-ring, and S a nonempty subset of N. The right ideal of N generated by S will be denoted by $R(S)$. If $H$ is a right ideal of N and $H = R(S)$ where S is a finite nonempty subset of N, then the right ideal H will be said to be finitely generated. In the case where S consists of the single element $\gamma$ of N, $R(S)$ will be denoted by $R(\gamma)$. Clearly if $S = \{\gamma_1, \ldots, \gamma_k\}$ where $k \geq 1$ is an integer, then $R(S) = R(\gamma_1) + R(\gamma_2) + \cdots + R(\gamma_k)$. The following theorem will be proved.

**Theorem 1.** Let V be a group and H a right ideal of $M_0(V)$. The right ideal H is finitely generated if, and only if, $H = (0 : L)$ where L is some nonempty subset of V. Furthermore if H is finitely generated then there exists $\gamma$ in H such that $H = R(\gamma)$.

This theorem will be proved with the aid of certain propositions and lemmas.

Let V be a group. Suppose $S_1$ and $S_2$ are subsets of V such that $S_1 \cup S_2 = V$ and $S_1 \cap S_2 = \{0\}$ then, as every function $\alpha$ of V into V that fixes zero can be expressed as a sum $\alpha_1 + \alpha_2$, where $\alpha_i$, $i = 1, 2$, are functions of V into V fixing zero and $\alpha_1$ is zero on $S_1$ and $\alpha_2$ is zero on $S_2$, the following proposition holds.

**Proposition 2.** Let V be a group and $S_1$ and $S_2$ subsets of V such that $S_1 \cup S_2 = V$, and $S_1 \cap S_2 = \{0\}$. It follows that $M_0(V) = (0 : S_1) \oplus (0 : S_2)$.

Suppose L is a nonempty subset of a group V. Let $L_1 = L \cup \{0\}$, and $L_2 = (V \setminus L) \cup \{0\}$. Clearly the subsets $L_1$ and $L_2$ of V satisfy the conditions of the above proposition and therefore, $(0 : L_1) \oplus (0 : L_2) = M_0(V)$. The identity 1 of $M_0(V)$ can therefore be written as $e_1 + e_2$ where $e_1$ is in $(0 : L_1)$, and $e_2$ is in $(0 : L_2)$. Also if $\alpha$ is in $M_0(V)$, then $e_1 \alpha + e_2 \alpha = \alpha$. Thus $e_1 M_0(V) = (0 : L_1)$, and $(0 : L_1)$ (= $(0 : L)$) is generated by the single element $e_1$ (if $L = \{0\}$, or $V$, then

Received by the editors May 31, 1979.

*AMS (MOS) subject classifications* (1970). Primary 16A76; Secondary 16A66, 16A42.

© 1980 American Mathematical Society
0002-9939/80/0000-0154/$01.50

475
(0 : L) is $M_0(V)$, or {0}, and the result holds with $e_1 = 1$, or 0). It now follows that right ideals of $M_0(V)$ of the form (0 : L) (L a nonempty subset of V) are finitely generated and the 'if' part of Theorem 1 is established.

If $\gamma$ is an element of $M_0(V)$ then we denote the set \{v \in V: v\gamma = 0\} by $Z(\gamma)$ (see [2, p. 200]).

**Lemma 3.** If V is a group and $\gamma$ an element of $M_0(V)$ then $R(\gamma) = (0 : Z(\gamma))$.

**Proof.** Let $\Lambda = (V \setminus Z(\gamma)) \cup \{0\}$. By Proposition 2, $M_0(V) = (0 : Z(\gamma)) \oplus (0 : \Lambda)$. Clearly $\gamma$ is in (0 : $Z(\gamma)$) and, $R(\gamma) < (0 : Z(\gamma))$. Now $1 = e_1 + e_2$, where $e_1$ is in (0 : $Z(\gamma)$), and $e_2$ is in (0 : $\Lambda$). Also, since, $e_2M_0(V) = (0 : \Lambda)$, $R(e_2) = (0 : \Lambda)$. Consider the element $\gamma + e_2$ of $M_0(V)$. If v is in ($V \setminus Z(\gamma)$) or (0), then ve_2 = 0 and v\gamma \neq 0, unless v = 0. If v is in $Z(\gamma)$, then since $Z(e_2) = (V \setminus Z(\gamma)) \cup \{0\}$, it follows that ve_2 \neq 0, unless v = 0. Thus if v in V is such that v(\gamma + e_2) = 0, then it must follow that v = 0. Now $R(\gamma + e_2) = M_0(V)$, or $R(\gamma + e_2)$ is contained in a maximal right ideal of $M_0(V)$. If $R(\gamma + e_2)$ is contained in a maximal right ideal of $M_0(V)$ by a result of M. Johnson [1] (see also Pilz [2, p. 200]) it follows that there exists v \neq 0 in V such that v(\gamma + e_2) = 0. Thus $R(\gamma + e_2) = M_0(V)$. Now $R(\gamma + e_2) < R(\gamma) + R(e_2)$, and since $R(\gamma) < (0 : Z(\gamma))$ and $R(e_2) = (0 : \Lambda)$, we conclude that $R(\gamma) \oplus (0 : \Lambda) = M_0(V)$. Clearly this can only happen if $R(\gamma) = (0 : Z(\gamma))$. The lemma is therefore proved.

**Corollary.** If $\lambda_1$ and $\lambda_2$ are elements of $M_0(V)$ such that $Z(\lambda_1) < Z(\lambda_2)$, then $R(\lambda_2) < R(\lambda_1)$.

By the above lemma, if the last statement of the theorem is proved the rest will follow.

**Lemma 4.** Let V be a group and H a finitely generated right ideal of $M_0(V)$. There exists $\gamma$ in H such that $R(\gamma) = H$.

**Proof.** Suppose P is a right ideal of $M_0(V)$ generated by a two element subset \{\gamma_1, \gamma_2\} of $M_0(V)$. If it is established that $P = R(\gamma')$ for some $\gamma'$ in $M_0(V)$, then the lemma will follow. Let $\Lambda = (V \setminus Z(\gamma_1)) \cup \{0\}$. We have, by Proposition 2, that $M_0(V) = (0 : Z(\gamma_1)) \oplus (0 : \Lambda)$, and by Lemma 3, $R(\gamma_1) = (0 : Z(\gamma_1))$. Now $\gamma_2 = \alpha + \beta$ where $\alpha$ is in (0 : $Z(\gamma_1)$), and $\beta$ is in (0 : $\Lambda$). Take $\gamma' = \gamma_1 + \beta$. Now every v in V is such that either $v\gamma_1 = 0$ or $v\beta = 0$. Thus if $v\gamma' = 0$, it follows that $v\gamma_1 + v\beta = 0$, and $v\gamma_1 = 0$. Hence $Z(\gamma') \subseteq Z(\gamma_1)$, and by the Corollary of Lemma 3, we conclude that $R(\gamma_1) < R(\gamma')$. Thus $-\gamma_1 + \gamma' = \beta$ is in $R(\gamma')$, and since $\alpha$ is in ($0 : Z(\gamma_1)$) = $R(\gamma_1)$, we see that $\alpha + \beta = \gamma_2$ is in $R(\gamma')$. Hence $R(\gamma') > R(\gamma_1) + R(\gamma_2) > P$. But $R(\gamma') < R(\gamma_1) + R(\beta)$ and, since $\alpha$ is in $R(\gamma_1)$, $R(\gamma') < R(\gamma_1) + R(\gamma_2)$ (= $P$). Thus $R(\gamma') = P$ and the lemma follows.

**References**


**Department of Mathematics, University of Auckland, Auckland, New Zealand**

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use