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# ALGEBRAIC CONDITIONS LEADING TO CONTINUOUS LATTICES

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ABSTRACT. This paper is concerned with sufficient conditions for a meet-continuous lattice $L$ to be a continuous lattice. In §2 this is shown to be true if the prime elements order generate and $(L, \bigvee)$ is a compact topological semilattice. In §3 it is shown that a meet-continuous lattice with finite breadth is a continuous lattice.

In 1972 D. Scott introduced a class of lattices called continuous lattices in order to provide models for the type free calculus in logic **[10]**. The discovery of K. Hofmann and A. Stralka **[2]** that continuous lattices are precisely the compact Hausdorff topological semilattices with identity and with a basis of subsemilattices (a class of semilattices previously studied by topological algebraists) has launched a recent flurry of activity in this area.

Proceeding from smaller classes of lattices to larger, we have the following hierarchy of lattices: (1) continuous lattices, (2) compact topological semilattices with 1, (3) meet-continuous lattices, (4) complete lattices. In this paper we address ourselves to the general question of finding sufficient conditions on a lattice in a larger class in order that it be in a smaller class. For example Hofmann and Stralka gave necessary and sufficient algebraic conditions on a complete lattice in order that it be a continuous lattice. In **[8]** it was shown that a compact topological semilattice with 1 on a finite-dimensional Peano continuum or on a totally disconnected space must have a basis of subsemilattices and hence be a continuous lattice. In **[7]** these results were generalized to a class of spaces including those which at each point are locally homeomorphic to the product of a totally disconnected space and a finite-dimensional Peano continuum. In **[6]** an example was constructed showing that class (1) is a proper class of class (2). Examples that the other inclusions are strict also exist.

**1. Definitions and basic results.** A lattice $L$ is *complete* if every subset has a least upper bound and a greatest lower bound. A set $D \subset L$ is up-directed if for any $d_1, d_2 \in D$, there exists $d_3 \in D$ such that $d_1 < d_3$ and $d_2 < d_3$. A semilattice $S$ is *meet-continuous* if whenever $x = \sup D$ for an up-directed set $D$, then $xy = \sup Dy$ for all $y \in D$. This definition is easily shown to be equivalent to the following conditions: if $\{x_\alpha\}$ in an increasing net (i.e. $\alpha < \beta$ implies $x_\alpha < x_\beta$) with supremum $x$ and $\{y_\beta\}$ is an increasing net with supremum $y$, then the increasing net $\{x_\alpha y_\beta\}$ has supremum $xy$ for all $x, y \in L$.

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Let $L$ be a complete lattice and define a new relation $\ll$ on $L$ as follows: $x \ll y$ if for all up-directed sets $D$ the relation $y \leqslant \sup D$ implies the existence of a $d \in D$ with $x \leqslant d$. Equivalently $x \ll y$ if $y \leqslant \sup A$ implies that $x \leqslant \sup F$ for some finite $F \subset A$. The lattice $L$ is *continuous* if for all $y \in L$, $y = \sup\{x\colon x \ll y\}$. By the results of Hofmann and Stralka [2] $L$ is continuous if and only if there is a compact Hausdorff topology on $L$ such that $L$ becomes a topological semilattice relative to the multiplication $(x, y) \to xy = \inf\{x, y\}$ and has a basis of neighborhoods at each point which are subsemilattices. In this case the topology is unique and is generated by the sets $\{s \in L\colon x \nleqslant s\}$ and $\{s \in L\colon x \ll s\}$, $x \in L$.

In a partially ordered set $P$, if $A \subset P$, then $\downarrow A = \{z \in P\colon z \leqslant a$ for some $a \in A\}$ and $\uparrow A = \{y \in P\colon a \leqslant y$ for some $a \in A\}$. The sets $\downarrow\{x\}$ and $\uparrow\{x\}$ are denoted by $\downarrow x$ and $\uparrow y$ resp. If $S$ is a (lower) semilattice, then a subset $A \subset S$ is said to *order generate* $S$ if $x = \inf(A \cap \uparrow x)$ for all $x \in S$.

An element $p \in S$, a semilattice, is *prime* if $xy \leqslant p$ implies $x \leqslant p$ or $y \leqslant p$. Let PRIME $S$ denote the set of all primes in $S$. By [1, 3.1] a continuous lattice $S$ is distributive if and only if PRIME $S$ is order generating.

**2. Distributive lattices.** In this section we consider how the assumption of continuity on one of the lattice operations affects the other lattice operation. It is well known that a compact topological semilattice is a meet-continuous semilattice (see e.g. [5]). Hence this is certainly a necessary condition for topological continuity.

2.1. PROPOSITION. *Let $L$ be a distributive complete lattice equipped with a compact Hausdorff topology for which $(L, \bigvee)$ is a topological semilattice. Then with respect to this topology $(L, \bigwedge)$ is a topological semilattice if and only if $L$ is a meet-continuous lattice.*

PROOF. We have already remarked that meet-continuity is necessary. To see that it is sufficient, let $x \in L$. Then $\downarrow x$ is a compact sublattice of $L$ (since $(L, \bigvee)$ is a topological semilattice and $\downarrow x = \{y\colon y \vee x = x\}$). Since $L$ is distributive the mapping $\lambda_x\colon L \to \downarrow x$ defined by $\lambda_x(y) = xy$ is a lattice homomorphism which always preserves arbitrary infs. The hypothesis of meet-continuity is precisely the condition needed for $\lambda_x$ to preserve sups of up-directed sets, i.e., $\lambda_x(\sup D) = \sup(\lambda_x D)$. Hence by [4, Theorem 15] the mapping $\lambda_x$ is continuous. Thus translations are continuous, i.e., multiplication in $(L, \bigwedge)$ is separately continuous. It follows from [3, Theorem 6] that multiplication is then jointly continuous. □

2.2. REMARK. The proposition may be proved in a more direct fashion with the stronger hypothesis that $(L, \bigvee)$ is a continuous lattice. In this case $\{x_\alpha\}$ converges to $x$ if and only if $x$ is the lim sup of $\{x_\alpha\}$ and all of its subnets. The hypotheses imply that $xy$ will be the lim sup for $\{x_\alpha y_\beta\}$ and all of its subnets and hence this net will converge to $xy$. However even in this case $(L, \bigwedge)$ need not be a continuous lattice.

I am indebted to M. Mislove for helping me work out the details of the proof of the following proposition.

2.3. PROPOSITION. *Let $L$ be a complete lattice equipped with a compact Hausdorff topology for which $(L, \bigvee)$ is a topological semilattice. If* PRIME $L$ *order generates $L$, then $L$ is a distributive continuous lattice (with respect to the meet operation).*

PROOF. Since PRIME $L$ order generates, $L$ is a distributive lattice. We show $L$ is meet-continuous. Let $D$ be a subset of $L$ which is directed upward, and let $x = \sup D$. For $y \in L$, we have $z = \sup(Dy) \leqslant xy$. Suppose $z < xy$. Then since PRIME $L$ order generates there exists $p \in$ PRIME $L$ such that $z \leqslant p$ and $xy \nleqslant p$. Since $xy \leqslant y$, we have $y \nleqslant p$. So $d \leqslant p$ since $dy \leqslant p$ for all $d \in D$. Hence $x = \sup D \leqslant p$, a contradiction since $xy \nleqslant p$. Thus $z = xy$ and $L$ is meet-continuous. It follows from Proposition 2.1 that $L$ is a compact topological lattice.

To show that $L$ is a continuous lattice, we must show that $x$ has a basis of subsemilattices at each point. Let $x \in L$. By **[8]** it suffices to show that $x$ has a basis of subsemilattices in the sublattice $\downarrow x$. Note that the primes of $\downarrow x$ order generate $\downarrow x$, since if $p \in$ PRIME $L$, then $xp$ is prime in $\downarrow x$. This reduction essentially amounts to allowing us to assume $x = 1$.

For each open set $U$ with $1 \in U$, let $z_U = \inf U$. Then the set $\{z_U\}$ is up-directed, hence converges to its supremum $y$ (since $L$ is compact). Since each $\uparrow z_U$ is a semilattice and a neighborhood of 1, if $y = 1$, then $\{\uparrow z_U\}$ would constitute a basis of subsemilattices at 1. We assume $y \neq 1$ and derive a contradiction.

Choose open sets $U_0$ and $V_0$ such that $y \in V_0 = \downarrow V_0$, $1 \in U_0 = \uparrow U_0$, and $\mathrm{cl}(U_0) \cap \mathrm{cl}(V_0) = \varnothing$. (We can do this since $L$ is a compact partially ordered space in the sense of Nachbin; hence $\downarrow y$ has a basis of open lower sets and 1 has a basis of open upper sets **[9]**.) Choose open sets $A_1$ and $B_1$ such that $1 \in A_1$, $y \in B_1$, $A_1 A_1 \subset U_0$, and $B_1 \vee B_1 \subset V_0$. Pick open sets $U_1$ and $V_1$ such that $1 \in U_1 = \uparrow U_1$, $y \in V_1 = \downarrow V_1$, $\mathrm{cl}(U_1) \subset \uparrow A_1$, $\mathrm{cl}(V_1) \subset \downarrow B_1$. Note that $U_1 U_1 \subset U_0$ and $V_1 \vee V_1 \subset V_0$. In this manner sequences of open sets $\{U_i\colon 0 \leqslant i\}$ and $\{V_i\colon 0 \leqslant i\}$ can be chosen recursively so that for each $i$, $1 \in U_i = \uparrow U_i$, $y \in V_i = \downarrow V_i$, $\mathrm{cl}(U_{i+1}) \subset U_i$, $\mathrm{cl}(V_{i+1}) \subset V_i$, and $U_{i+1} U_{i+1} \subset U_i$, $V_{i+1} \vee V_{i+1} \subset V_i$.

Let $z_i = \inf U_i$. By the choice of $y$, $z_i \leqslant y$, and hence $z_i \in V_i$. Since $z_i$ is in the closure of the subsemilattice generated by $U_i$, we can choose $y_i \in V_i$ such that $y_i = \bigwedge F$ for some finite $F \subset U_i$. By an argument which is almost standard by now (see the proof of Theorem 8 of **[4]** or Proposition 6.3 of **[5]**) we conclude that $w = \sup\{y_i\colon 1 \leqslant i\} \in \mathrm{cl}(V_0)$. Also by the way the sequence $\{U_i\}$ was chosen, $T = \bigcap U_i = \bigcap \mathrm{cl}(U_i)$ is a compact subsemilattice, and hence has a least element $t$. Since $U_0 = \uparrow U_0$, $\uparrow t \subset U_0$. Thus $w \notin \uparrow t$. Since the primes order generate, there exists a prime $p$ such that $w \leqslant p$ and $t \nleqslant p$. Since $t \notin \downarrow p$, $T \cap \downarrow p = \varnothing$. Thus there exists an $i$ such that $\mathrm{cl}(U_i) \cap \downarrow p = \varnothing$. But there exists a finite set $F$ such that $F \subset U_i$ and $\inf F = y_i \leqslant w \leqslant p$. Since $p$ is prime $x \leqslant p$ for some $x \in F$, i.e., $U_i \cap \downarrow p \neq \varnothing$, a contradiction. $\square$

The distinction between complete lattices in which $(L, \bigwedge)$ is a compact topological semilattice and continuous lattices appears to be a rather fine one. Examples of the former that are not the latter have been few and far between (see **[6]**). The preceding proposition may be restated to give another set of sufficient conditions for a compact semilattice to be a continuous lattice.

2.4. COROLLARY. *Let $L$ be a complete lattice equipped with a compact Hausdorff topology for which $(L, \wedge)$ is a topological semilattice. If* PRIME $L$ *order generates and $(L, \vee)$ is join-continuous, then $L$ is a continuous lattice and a topological lattice.*

PROOF. Since PRIME $L$ order generates, $L$ is distributive. By 2.1 (with the role of the meet and join operations interchanged) $L$ is a compact topological lattice. Hence by 2.3 $L$ is a continuous lattice.

REMARK. By [4] the topology on $L$ is an intrinsic topology, the so-called $CO$-topology.

In [1] it is shown that a continuous lattice $L$ which is distributive is order generated by PRIME $L$. It is unknown whether the following converse holds: Let $L$ be a compact topological semilattice with 1 in which PRIME $L$ generates. Then $L$ is continuous. Corollary 2.4 shows this conjecture is true with the additional assumption that $(L, \vee)$ is join-continuous.

Let $X$ be an infinite set and let $L = X \cup \{0, 1\}$. Define a lattice structure on $L$ by $xy = 0$, $x \vee y = 1$, for $x, y \in X$, $x \neq y$. Then $L$ is a complete lattice which is a continuous lattice with respect to both of its operations. However the $CL$-topologies induced by each of the operations do not agree ($\{0\}$ is open for $(L, \vee)$ while $\{1\}$ is open for $(L, \wedge)$). By 2.1 and the uniqueness of the topology [4] the two must agree for distributive lattices. Are there other classes of interest for which they would agree?

**3. Semilattices of finite breadth.** A subset $A$ of a semilattice $S$ is said to be an *irredundant set* if for any two finite subsets $F_1, F_2 \subset P$, $\inf F_1 = \inf F_2$ implies $F_1 = F_2$. Let ${}^N2$ denote the semilattice of all finite subsets of $N$, the natural numbers, under union. The singletons in ${}^N2$ form an irredundant set, and it is a straightforward exercise to show that a semilattice has a countable irredundant set if and only if it has a semilattice isomorphic to ${}^N2$.

The semilattice $S$ has *finite breadth* $n$ if $n$ is the largest cardinal such that $S$ has an irredundant subset of cardinality $n$; $S$ is said to have *weak finite breadth* if it has no countable irredundant subset (or equivalently no isomorphic copy of ${}^N2$). Finite breadth implies weak finite breadth, but not conversely.

3.1. PROPOSITION. *If $S$ is a complete meet-continuous lattice, $x \nleqslant y$, and* $\sup\{z: z \ll x\} \leqslant y$, *then $\downarrow x \setminus \downarrow y$ contains a countable irredundant set.*

PROOF. First of all note that as a result of meet continuity $a \ll b$ iff $\sup D = b$ for some up-directed set implies $a \leqslant d$ for some $d \in D$.

Since not $x \ll x$, there exists a directed set $D$ with $x = \sup D$, but $x \neq d$ for all $d \in D$. Pick $x_1 \in D$ such that $x_1 \nleqslant y$.

Suppose $A_K = \{x_1, \ldots, x_K\}$ has been chosen satisfying (i) $A_K$ is irredundant, and (ii) the subsemilattice $S_K$ generated by $A_K$ is a subset of $\downarrow x \setminus \downarrow y$. Let $z = x_1 x_2 \cdots x_K$. Since not $z \ll x$, there exists a directed set $D$ with $x = \sup D$, but $z \nleqslant d$ for all $d \in D$. For each $s \in S_K$, $sD$ is a directed set converging up to $s$. Since $S_K$ is finite, there exists $b \in D$ such that $sd \neq td$ if $s \neq t$ for all $s, t \in S_K$, for all $d \geqslant b$. Since $\sup zD = z$ and $z \nleqslant y$, there exists $p \geqslant b$, $p \in D$, such that

$pz \nleqslant y$. Let $x_{K+1} = p$. Then it is easily verified that $\{x_1, \ldots, x_{K+1}\}$ is irredundant and the subsemilattice this set generates is a subset of $\downarrow x \setminus \downarrow y$. Hence by recursion there exists a set with the desired properties. □

3.2. COROLLARY. *Let $S$ be a complete meet-continuous lattice satisfying the following condition: for all $x \in S$, $x = \sup\{y \leqslant x: \downarrow y$ has weak finite breadth$\}$. Then $S$ is a continuous lattice. In particular, a complete lattice of finite breadth is continuous if and only if it is meet-continuous.*

PROOF. Let $x \in S$. Let $y = \sup\{z: z \ll x\}$. If $y < x$, then there exists $q \leqslant x$ such that $\downarrow q$ has weak finite breadth and $q \nleqslant y$. Let $p = \sup\{t: t \ll q\}$. Since $t \ll q$ implies $t \ll x$, we have $p \leqslant y$. Hence $p < q$. Now by 3.1, $\downarrow q \setminus \downarrow p$ contains a countable irredundant set. However, this is impossible since $\downarrow q$ has weak finite breadth. Hence $x = y$, and thus $S$ is continuous. The second statement is immediate. □

3.3. COROLLARY. *Let $L$ be a complete lattice of weak finite breadth. Then $L$ is a distributive continuous lattice if and only if* PRIME $L$ *order generates $L$.*

PROOF. By [1] if $L$ is a distributive continuous lattice, then PRIME $L$ order generates $L$. Conversely, if PRIME $L$ order generates, then $L$ is a meet-continuous lattice (see the proof of Proposition 2.3). Hence by 3.2 $L$ is a continuous lattice. The order generation of PRIME $L$ implies that $L$ is distributive. □

Semilattices and lattices of finite breadth have played a major role in the study of topological semilattices and lattices (since they have close connections to the finite-dimensional ones). It is somewhat striking that such semilattices have the rather simple algebraic description of being complete and meet-continuous.

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