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THE NONEXISTENCE OF INVARIANT UNIVERSAL MEASURES ON SEMIGROUPS

V. KANNAN AND S. RADHAKRISHNESWARA RAJU

ABSTRACT. We prove that if S is an uncountable subsemigroup of a group, then every (left or right)-translation invariant σ-finite measure defined on all subsets of S must be trivial. This answers a question posed by Ryll-Nardzewski and Telgarsky.

A universal measure on a set is, by definition, a (countably-additive, positive, extended real-valued) measure defined on all subsets of that set. A measure μ on (X, Σ) is said to be semiregular, if whenever $A \in \Sigma$ and $\mu(A) > 0$, there is $B \in \Sigma$ such that $B \subset A$ and such that $0 < \mu(B) < \infty$. It is easily seen that every σ-finite measure is semiregular. We start with a Proposition that will be heavily used in our Theorem. $\aleph_1$ denotes the first uncountable cardinal number.

PROPOSITION. *Every universal semiregular measure is $\aleph_1$-additive.*

PROOF. Let us recall the definition of $\aleph_1$-additivity. This means that whenever $\{A_\alpha: \alpha \in J\}$ is a class of pairwise disjoint measurable sets and $|J| = \aleph_1$ and if $A = \bigcup \{A_\alpha: \alpha \in J\}$ is measurable, then it is true that the measure of A is equal to the sum of the measures of A_α's. If μ is the measure, what we demand is $\mu(A) = \sum_{\alpha \in J} \mu(A_\alpha)$, the sum on the right being defined in the most natural way, as

$$\operatorname{Sup}\left\{ \sum_{\alpha \in F} \mu(A_\alpha) : F \text{ is a finite subset of } J \right\}.$$

To prove the Proposition, let μ be a universal semiregular measure on a set X, let J be an index set with cardinality $\aleph_1$, let $\{A_\alpha: \alpha \in J\}$ be a family of pairwise disjoint subsets of X indexed by J and let A be their union. We have to prove that

$$\mu(A) = \sum_{\alpha \in J} \mu(A_\alpha). \tag{1}$$

Case 1. Let $\mu(A_\alpha) = 0$ for every $\alpha \in J$. Then we claim that $\mu(A) = 0$. If not, by the semiregularity of μ, there is some $B \subset A$ such that $0 < \mu(B) < \infty$. Define a measure ν on the index set J by the rule

$$\nu(E) = \mu\left(\bigcup_{\alpha \in E} B \cap A_\alpha \right)$$

It is easily checked that ν is also countably additive. In fact it is a universal measure on J satisfying $\nu(J) = \mu(B)$ and hence $0 < \nu(J) < \infty$. Further if $\alpha \in J$ is

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any element, we have

$$\nu(\{\alpha\}) = \mu(B \cap A_\alpha) \leq \mu(A_\alpha) = 0.$$

Since J is of cardinality $\aleph_1$, this contradicts a well-known theorem of Ulam (see [**O**, Theorem 5.6, p. 25]). This contradiction proves that $\mu(A)$ should be zero.

Case 2. Let $\mu(A_\alpha) > 0$ for uncountably many α in J. Then $\sum_{\alpha \in J} \mu(A_\alpha)$ has to be ∞. Further, there is a positive integer n such that $\mu(A_\alpha) > 1/n$ for infinitely many (in fact, uncountably many) α in J. Since A contains all these A_α's, the countable additivity of μ implies that $\mu(A)$ is also ∞. Thus the equality (1) is valid in this case also.

Case 3. Let $J_1 = \{\alpha \in J\colon\ \mu(A_\alpha) > 0\}$ and let J_1 be countable. Let $B = \bigcup_{\alpha \in J_1} A_\alpha$. Then we have

$$\begin{aligned}
\mu(A) &= \mu(B) + \mu(A \setminus B) \\
&= \sum_{\alpha \in J_1} \mu(A_\alpha) + \mu(A \setminus B) \quad \text{by countable additivity} \\
&= \sum_{\alpha \in J_1} \mu(A_\alpha) + 0 \quad \text{by Case 1, since} \\
&\qquad\qquad A \setminus B = \bigcup \{A_\alpha\colon \alpha \in J \setminus J_1\} \\
&\qquad\qquad \text{and since } \mu(A_\alpha) = 0\ \forall \alpha \in J \setminus J_1 \\
&= \sum_{\alpha \in J} \mu(A_\alpha) \quad \text{since } \mu(A_\alpha) = 0\ \forall \alpha \in J \setminus J_1.
\end{aligned}$$

Thus the Proposition is proved.

THEOREM. *Let S be an uncountable semigroup embeddable in a group. Let μ be a σ-finite universal right translation-invariant measure on S. Then $\mu = 0$.*

PROOF. Let G be a group in which S is embedded as a subsemigroup. Let E be any subset of S having cardinality $\aleph_1$. Let H be the subgroup of G generated by E. Let A be a subset of G meeting each left coset of H in G, in exactly one point. Then one easily verifies that Ax and Ay are disjoint, whenever x and y are distinct elements of H. Let

$$A_x = (Ax) \cap S \tag{2}$$

for every x in H. Then we have

$$S = \bigcup \{A_x\colon x \in H\} \tag{3}$$

because we have $G = \bigcup \{Ax\colon\ x \in H\}$. Thus (3) represents S as the union of a class of pairwise disjoint sets, indexed by the set H having cardinality $\aleph_1$. Since μ is σ-finite and hence semiregular, the previous Proposition applies. Thus, we have

$$\mu(S) = \sum_{x \in H} \mu(A_x). \tag{4}$$

Now consider two cases.

Case 1. Let $\mu(A_x) = 0$ for every x in H. Then by (4) we have $\mu(S) = 0$ and thus the result is proved in this case.

Case 2. Let $\mu(A_x) > 0$ for some x in H. Now if y is any element of E, we have

$$\begin{aligned} A_x y &= (Ax \cap S)y = (Axy) \cap Sy \\ &\subset Axy \cap S \quad \text{since } S \text{ is closed under multiplication and } y \in S \\ &= A_{xy} \end{aligned}$$

and therefore $\mu(A_{xy}) \geqslant \mu(A_x y) = \mu(A_x)$ because μ is translation-invariant, > 0 by our assumption in this case. Thus $\{A_{xy}: y \in E\}$ is a collection of pairwise disjoint subsets of S indexed by a set of cardinality $\aleph_1$, such that every member in this collection has positive measure. This contradicts the assumption that μ is σ-finite. Hence Case 2 does not arise at all.

COROLLARY. *Let S be an uncountable commutative cancellative semigroup. Then every σ-finite translation-invariant universal measure on S is trivial.*

PROOF. Every such semigroup can be embedded in a group and therefore our Theorem applies.

REMARKS. The above Corollary answers a question posed in **[R-T]**. The special case of the above Theorem, where S itself is assumed to be a group, has been proved first in **[E-M]** and then by a different method in **[R-T]**.

We conclude with the following open question.

Problem. Is every translation-invariant universal semiregular measure on a group necessarily a multiple of the counting measure?

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UNIVERSITY OF HYDERABAD, NAMPALLY STATION ROAD, HYDERABAD 500 001, INDIA