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AN ASYMPTOTIC FORMULA FOR THE TAYLOR COEFFICIENTS OF AUTOMORPHIC FORMS

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ABSTRACT. An asymptotic estimate for the lattice of a Fuchsian group with quotient of finite area is discussed. The estimate is used to obtain an asymptotic formula for the Taylor coefficients of holomorphic automorphic forms.

Let G be a Fuchsian group acting on the unit disc U with the quotient U/G of finite Poincaré area. S. J. Patterson [5] has given an asymptotic estimate for the lattice arising from G. The estimate enables us to give an asymptotic formula for the integral of an automorphic function over compact subdiscs of U. As an application we obtain an asymptotic formula for the Taylor coefficients of automorphic holomorphic forms. The reader is referred to the article of J. Lehner as a general reference [4].

We assume throughout that G is a Fuchsian group with U/G of finite Poincaré area. A function f defined in U is a G automorphic q form if $f \circ \sigma\, \sigma'^q = f$ for all $\sigma \in G$ and q an integer. Denote the space of holomorphic automorphic cusp q forms, $q \geqslant 1$, by $A_q(G)$ [4]. Fix a measurable fundamental domain Ω for G. A norm for $A_q(G)$ is defined by setting

$$\|\psi\|^2 = \int_\Omega |\psi|^2(1 - |z|^2)^{2q-2}dx\, dy$$

for $\psi \in A_q(G)$. A Hilbert space $L^2(G)$ of 0 forms is defined in terms of the norm

$$\|f\|_0^2 = \int_\Omega |f|^2(1 - |z|^2)^{-2}dx\, dy.$$

The norms are independent of the choice of Ω. Denote by $V(G)$ the Poincaré area $\int_\Omega 4(1 - |z|^2)^{-2}dx\, dy$ of Ω. The lattice estimate we require was given by Patterson [5]. His discussion will be summarized so that we can give a refinement of the error estimate.

A continuous point pair invariant is a function with domain $U \times U$ such that

$$k(z_1, z_2) = k(2 \cosh d(z_1, z_2) - 2)$$

where $d(\,,\,)$ is the Poincaré distance. Define the auxiliary function

$$L(z_1, z_2) = 2 \cosh d(z_1, z_2) + 2.$$

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Following Selberg, we associate to k three transforms Q, h, and g. The relations among these are given by the formulae

$$Q(w) = \int_w^\infty k(t)(t-w)^{-1/2}\,dt, \qquad k(t) = -\frac{1}{\pi}\int_t^\infty (w-t)^{-1/2}\,dQ(w),$$

$$Q(e^u + e^{-u} - 2) = g(u),$$

$$h(r) = \int_{-\infty}^\infty e^{iru}g(u)\,du, \qquad g(u) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-iru}h(r)\,dr.$$

It is customary to describe the hypotheses in terms of h:

(i) $h(r) = h(-r)$,

(ii) for some $\varepsilon > 0$, $h(r)$ is analytic in $|\mathrm{Im}\, r| < \frac{1}{2} + \varepsilon$, and

(iii) in this region $h(r) = O((1 + |r|^2)^{-1-\varepsilon})$.

The Poincaré series

$$K(z_1, z_2) = \sum_{\sigma \in G} k(L(\sigma z_1, z_2) - 4)$$

converges uniformly and absolutely on compact sets. Let $\{\varphi_\mu\}$ be a complete orthonormal (relative to $L^2(G)$) system of automorphic eigenfunctions for the operator $\frac{1}{4}(1 - |z|^2)^2\,(\partial^2/\partial x^2 + \partial^2/\partial y^2)$. Denote the eigenvalue of φ_μ by $-\lambda_\mu$, $\lambda_\mu \geqslant 0$. A. Selberg has given the expansion

$$K(z_1, z_2) = \sum_\mu h\Big(\big(\lambda_\mu - \tfrac{1}{4}\big)^{1/2}\Big)\varphi_\mu(z_1)\varphi_\mu(z_2) + \sum_{p \in P} \frac{\pi}{4}\int_{-\infty}^\infty h(r)E_p\big(z_1, \tfrac{1}{2} + ir\big)E_p\big(z_2, \tfrac{1}{2} - ir\big)\,dr, \tag{1}$$

where P is a complete set of inequivalent cusps and $E_p(z, s)$ is the Eisenstein series for the cusp $p \in P$ [7]. As an example, consider $h(r) = (r^2 + u^2)^{-\sigma}$ where $u > 1/4$, $\sigma > 1$. The appropriate hypotheses are satisfied. The inequality $h((\lambda_\mu - 1/4)^{1/2}) > 0$ holds. The series and integral of (1) converge absolutely by Mercer's theorem [1, p. 138] for $z_1 = z_2$ the series represents an element of $L^2(G)$. Patterson introduces the kernel

$$k_X(t) = \begin{cases} 1 - t/X, & t \leqslant X, \\ 0, & t > X. \end{cases}$$

The transform $h_X(r)$ is obtained explicitly in terms of the Euler gamma function and the hypergeometric function. Let r_μ be chosen such that $\lambda_\mu = 1/4 + r_\mu^2$ and for $s_\mu = \frac{1}{2} + ir_\mu$, $\mathrm{Re}(s_\mu) > \frac{1}{2}$ when $\lambda_\mu < 1/4$. Patterson demonstrates the following.

PROPOSITION 1. *Suppose that $X > \frac{1}{2}$. Then $h_X(r)$ is analytic in $|\mathrm{Im}(r)| < 3/4$ and satisfies, for some $c_1 > 0$, the inequality*

$$|h_X(r)| \leqslant c_1 X^{\mathrm{Re}(1/2+ir)}(1 + |r|^2)^{-5/4}.$$

Furthermore, if $\mathrm{Re}(ir) > 0$, then, for fixed r,

$$h_X(r) = \pi^{1/2}\big(\Gamma(ir)/\Gamma\big(\tfrac{5}{2} + ir\big)\big)X^{1/2+ir} + O(X^{|\mathrm{Re}(1/2-ir)|}).$$

We use this to estimate the series occuring in (1)

$$\left|\sum_{\lambda_\mu > 1/4} h_X(r_\mu)\varphi_\mu(z_1)\varphi_\mu(z_2)\right| \leq \sum_{\lambda_\mu > 1/4} |h_X(r_\mu)||\varphi_\mu(z_1)\varphi_\mu(z_2)|$$
$$\leq c_1 X^{1/2} \sum_{\lambda_\mu > 1/4} \left(1 + |r_\mu|^2\right)^{-5/4} |\varphi_\mu(z_1)\varphi_\mu(z_2)|$$

where the last series converges in $L^2(U/G \times U/G)$ by the Cauchy-Schwarz inequality and, since $h_1(r) = (1 + r^2)^{-5/4}$, satisfies the hypotheses (i), (ii) and (iii). The Cauchy-Schwarz inequality yields

$$\int_{-\infty}^{\infty} |h_X(r)E_p\left(z_1, \tfrac{1}{2} + ir\right)E_p\left(z_2, \tfrac{1}{2} - ir\right)| dr$$
$$\leq \left(\int_{-\infty}^{\infty} |h_X(r)|\, |E_p\left(z_1, \tfrac{1}{2} + ir\right)|^2\, dr\right)^{1/2} \left(\int_{-\infty}^{\infty} |h_X(r)|\, |E_p\left(z_2, \tfrac{1}{2} - ir\right)|^2\, dr\right)^{1/2}$$

and from Proposition 1 this is bounded by $X^{1/2}\varepsilon_1(z_1, z_2)$ with

$$\varepsilon_1(z_1, z_2) = c_1\left(\int_{-\infty}^{\infty} h_1(r)|E_p\left(z_1, \tfrac{1}{2} + ir\right)|^2\, dr\right)^{1/2} \times \left(\int_{-\infty}^{\infty} h_1(r)|E_p\left(z_2, \tfrac{1}{2} - ir\right)|^2\, dr\right)^{1/2}. \tag{2}$$

Combining these remarks we have

$$K(z_1, z_2) = \left(\frac{\pi}{2V(G)}\right)X + \sum_{1/2 < s_\mu < 1} \pi^{1/2} \frac{\Gamma\left(s_\mu - \frac{1}{2}\right)}{\Gamma\left(s_\mu + \frac{1}{2}\right)} X^{s_\mu}\varphi_\mu(z_1)\varphi_\mu(z_2)$$
$$+ X^{1/2}(\varepsilon_1(z_1, z_2) + \varepsilon_2(z_1, z_2))$$

with Γ the Euler gamma function, ε_1, as above and ε_2 the bound for the omitted terms of the series (we abuse notation by replacing quantities by their bounds). The estimate $\varepsilon_2 \in L^2(U/G \times U/G)$ is valid. We postpone the consideration of ε_1.

Define the counting function $N(X; z_1, z_2)$ to be the number of $\sigma \in G$ such that $L(\sigma z_1, z_2) \leq X$. As an excerise we derive Patterson's asymptotic estimate for $N(X; z_1, z_2)$. Define

$$N_1(X; z_1, z_2) = \sum_\sigma k_X(L(\sigma z_1, z_2) - 4)$$

and observe that

$$F(X) = XN_1(X; z_1, z_2) = \int_4^{4+X} N(v; z_1, z_2)\, dv.$$

The approach consists of estimating the difference quotients

$$(F(X \pm X^{3/4}) - F(X))/\pm X^{3/4}.$$

Observing that N is increasing in X, one has that

$$\frac{F(X - X^{3/4}) - F(X)}{-X^{3/4}} \leq N(X; z_1, z_2) \leq \frac{F(X + X^{3/4}) - F(X)}{X^{3/4}}.$$

A short argument now yields the formula

$$N(X; z_1, z_2) = \left(\frac{\pi}{V(G)}\right)X + \sum_{3/4<s_\mu<1} \pi^{1/2}\frac{\Gamma(s_\mu - \frac{1}{2})}{\Gamma(s_\mu + 1)} X^{s_\mu}\varphi_\mu(z_1)\varphi_\mu(z_2) + c_2X^{3/4}(\varepsilon_1(z_1, z_2) + \varepsilon_2(z_1, z_2)) \tag{3}$$

for $c_2 > 0$ an appropriate constant. The relation $L(0, R) = 4/(1 - R^2)$ is required. A continuous automorphic function $f(z)$ is a cusp function if $|f(z)|/(1 - |z|^2)$ converges to 0 as z approaches nontangentially a cusp of G on ∂U. The expression $|f(z)|/(1 - |z|^2)$ projects to a function defined in a neighborhood of the cusps of U/G. A cusp function is necessarily bounded.

THEOREM 2. *Let U/G be of finite Poincaré area and f an automorphic cusp function. Then*

$$\int_{|z|<R} f(1 - |z|^2)^{-2}dx\,dy = \frac{4\pi}{V(G)(1 - R^2)} \int_\Omega f(1 - |z|^2)^{-2}dx\,dy + \sum_{3/4<s_\mu<1} \pi^{1/2}\frac{\Gamma(s_\mu - \frac{1}{2})}{\Gamma(s_\mu + 1)}\left(\frac{4}{1 - R^2}\right)^{s_\mu}\phi_\mu(0) \times \int_\Omega \phi_\mu f(1 - |z|^2)^{-2}dx\,dy + O(X^{3/4}).$$

PROOF. By definition

$$\int_{|z|<R} f(1 - |z|^2)^{-2}dx\,dy = \int_\Omega N\left(\frac{4}{1 - R^2}; 0, z\right)f(1 - |z|^2)^{-2}dx\,dy.$$

Define

$$\alpha_0(f) = \frac{4\pi}{V(G)} \int_\Omega f(1 - |z|^2)^{-2}dx\,dy,$$

$$\alpha_\mu(f) = \pi^{1/2}4^{s_\mu}\phi_\mu(0)\frac{\Gamma(s_\mu - \frac{1}{2})}{\Gamma(s_\mu + 1)} \int_\Omega \phi_\mu f(1 - |z|^2)^{-2}dx\,dy. \tag{4}$$

Then from (3)

$$\int_{|z|<R} f(1 - |z|^2)^{-2}dx\,dy = \frac{\alpha_0(f)}{(1 - R^2)} + \sum_{3/4<s_\mu<1} \frac{\alpha_\mu(f)}{(1 - R^2)^{s_\mu}} + c_2X^{3/4}\int_\Omega(\varepsilon_1(0, z) + \varepsilon_2(0, z))|f|(1 - |z|^2)^{-2}dx\,dy.$$

As $\varepsilon_2 \in L^2$, the integral $\int_\Omega|\varepsilon_2(0, z)f|(1 - |z|^2)^{-2}dx\,dy$ is finite. It remains to estimate the integral involving ε_1. Let σ_p be a Möbius transformation mapping U to the upper half-plane H with $\sigma_p(p) = \infty$. The element σ_p is chosen such that the stabilizer of ∞ in $\sigma_p G\sigma_p^{-1}$ is generated by the transformation $z \to z + 1$. Let Ω be chosen such that $\sigma_p(\Omega) \cap \{z|\operatorname{Im} z > 1\}$ is the vertical strip $\{z|\operatorname{Im} z > 1, 0 < \operatorname{Re} z < 1\}$. We must estimate the integrals

$$\int_{\sigma_p(\Omega)} \left(\int_{-\infty}^{\infty} h_1(r) |E_p(z, \tfrac{1}{2} + ir)|^2 \, dr \right)^{1/2} \frac{|f(z)|}{y^2} \, dx \, dy, \tag{5}$$

$p \in P$. T. Kubota gives the following expansion

$$\int_{\sigma_p(\Omega) \cap \{\operatorname{Im} z < Y\}} \int_{-\infty}^{\infty} h_1(r) \frac{|E_p(z, \tfrac{1}{2} + ir)|^2}{y^2} \, dr \, dx \, dy$$
$$= g_1(0) \log Y + a_p + O(1), \tag{6}$$

$Y > 1$, where $g_1(u)$ is the Fourier transform of $h_1(r)$ and a_p is an explicit constant, [**2**, p. 107]. A cusp function f satisfies $|f(z)|y \to 0$ as $\operatorname{Im} z \to \infty$ in $\sigma_p(\Omega)$. We observe first that f is bounded on $\sigma_p(\Omega)$; hence the integral (5) over the domain $\sigma_p(\Omega) \cap \{\operatorname{Im} z \le 1\}$ is finite. The integral (6) over only the domain $\{z \in \sigma_p(\Omega) | 1 \le Y_1 \le \operatorname{Im} z \le Y_2\}$ is given by $g_1(0) \log Y_2/Y_1 + O(1)$. By hypothesis, $|f(z)|y$ is bounded. Partition the domain $\sigma_p(\Omega) \cap \{\operatorname{Im} z > 1\}$ into the subdomains $\sigma_p(\Omega) \cap \{e^n \le \operatorname{Im} z \le e^{n+1}\}$, $n \in \mathbf{Z}^+$. The integral (5) over such a domain is bounded by Me^{-n} for an appropriate constant M. Hence the integral (5) is convergent, the desired conclusion.

Given $\psi \in A_q(G)$ we observe that $|\psi|^2(1 - |z|^2)^{2q}$ is a cusp function. Define

$$\alpha_j = \alpha_j\left(|\psi|^2(1 - |z|^2)^{2q}\right)$$

for appropriate j.

THEOREM 3. *Let $\psi \in A_q(G)$, $q > 1$, be given in U as the Taylor series $\sum_n a_n z^n$. Then*

$$\pi \sum_{n=0}^{\infty} \frac{|a_n|^2 T^{n+1}}{n+1} = \frac{\alpha_0}{(2q-1)(1-T)^{2q-1}}$$
$$+ \sum_{3/4 < s_\mu < 1} \frac{s_\mu \alpha_\mu}{(2q-2+s_\mu)(1-T)^{2q-2+s_\mu}}$$
$$+ O\left(\frac{1}{(1-T)^{2q-2+3/4}} \right) \tag{7}$$

and

$$\pi \sum_{n=0}^{N} \frac{|a_n|^2}{n+1} = \frac{\alpha_0 N^{2q-1}}{\Gamma(2q)(2q-1)}$$
$$+ \sum_{3/4 < s_\mu < 1} \frac{s_\mu \alpha_\mu N^{2q-2+s_\mu}}{\Gamma(2q-1+s_\mu)(2q-2+s_\mu)}$$
$$+ O\left(\frac{N^{2q-2+\delta}}{\log N} \right) \tag{8}$$

for δ satisfying $3/4 < \delta < s_\mu$.

PROOF. Define

$$B(R^2, n, q) = \int_0^R r^{2n+1}(1 - r^2)^{2q-2}\, dr.$$

Now from Theorem 2

$$2\pi \sum_n |a_n|^2 B(R^2, n, q) = \int_{|z|<R} |\psi|^2 (1 - |z|^2)^{2q-2} dx\, dy$$

$$= \frac{\alpha_0}{1 - R^2} + \sum_{3/4<s_\mu<1} \frac{\alpha_\mu}{(1 - R^2)^{s_\mu}} + O\left(\frac{1}{(1 - R^2)^{3/4}}\right).$$

By inspection $O(1/(1 - R^2)^{3/4})$ refers to a function ε, real analytic in R^2, satisfying $|\varepsilon(R^2)| < c/(1 - R^2)^{3/4}$. We substitue $T = R^2$ and differentiate to obtain

$$\pi \sum_n |a_n|^2 T^n (1 - T)^{2q-2} = \frac{\alpha_0}{(1 - T)^2} + \sum_{3/4<s_\mu<1} \frac{s_\mu \alpha_\mu}{(1 - T)^{s_\mu+1}} + \varepsilon'(T);$$

dividing by $(1 - T)^{2q-2}$ we have

$$\pi \sum_n |a_n|^2 T^n = \frac{\alpha_0}{(1 - T)^{2q}} + \sum_{3/4<s_\mu<1} \frac{s_\mu \alpha_\mu}{(1 - T)^{2q-1+s_\mu}} + \frac{\varepsilon'(T)}{(1 - T)^{2q-2}}.$$

We form the indefinite integral of each side to obtain

$$\pi \sum_n \frac{|a_n|^2 T^{n+1}}{n + 1} = \frac{\alpha_0}{(2q - 1)(1 - T)^{2q-1}}$$

$$+ \sum_{3/4<s_\mu<1} \frac{s_\mu \alpha_\mu}{(2q - 2 + s_\mu)(1 - T)^{2q-2+s_\mu}} + O\left(\frac{1}{(1 - T)^{2q-2+3/4}}\right),$$

where the error estimate is obtained by an integration by parts. On substituting $T = e^{-t}$ the formula is alternately given as

$$\pi \sum_n \frac{|a_n|^2 e^{-t(n+1)}}{n + 1} = \frac{\alpha_0}{(2q - 1)t^{2q-1}}$$

$$+ \sum_{3/4<s_\mu<1} \frac{s_\mu \alpha_\mu}{(2q - 2 + s_\mu)t^{2q-2+s_\mu}} + O\left(\frac{1}{t^{2q-2+3/4}}\right).$$

The hypothesis of Patterson's Tauberian theorem are now satisfied and the conclusion follows [6].

We note that formulas similar to the above for $q = 1$ appeared in [5]. The origin of U can be made to correspond to an arbitrary point of the Riemann surface U/G be replacing G by a conjugate. The resulting effect on the Taylor coefficients of ψ is described by the factor $\phi_\mu(0)$ appearing in the expression for α_μ (see (4)).

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