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A SIMPLE C^*-ALGEBRA WITH NO NONTRIVIAL PROJECTIONS

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ABSTRACT. A C^*-algebra is constructed which is separable, simple, nuclear, nonunital, and contains no nonzero projections. Some results on automorphisms of AF algebras are also obtained.

A C^*-algebra is said to be *projectionless* if it contains no projections other than 1 (if present) and 0. It has long been an open question whether there exists a projectionless simple C^*-algebra (see [**13**, p. 18], [**6**, 1.9.6], [**8**, p. 81], [**14**, p. 242]). In this paper we construct a projectionless simple separable nuclear nonunital C^*-algebra.

It is quite possible that the methods of this paper can be modified to yield a projectionless simple unital C^*-algebra. It is conjectured that the C^*-algebra generated by the regular representation of the free group on two generators (known to be simple and unital) is projectionless.

1. Outline of construction. The general method of construction is motivated by the construction of the Bunce-Deddens weighted shift algebras [**5**] as described by Green [**12**, p. 248]. The algebra A is constructed as an inductive limit of C^*-algebra A_n, each of which is a continuous field algebra on a circle T with a constant simple fiber B with the embedding ϕ_n: $A_n \to A_{n+1}$ inducing the "twice around" map $z \to z^2$ of T onto T.

The algebra B will be the (unique) simple unital AF algebra whose ordered group $K_0(B)$ is isomorphic to the additive group of real algebraic numbers [**10**, 2.2]. B has the following properties:

(1) B has a unique normalized trace τ, which is faithful.

(2) If p and q are projections in B, then $p \sim q$ if and only if $\tau(p) = \tau(q)$.

(3) If λ is any algebraic number with $0 \leqslant \lambda \leqslant 1$, then there is a projection $p \in B$ with $\tau(p) = \lambda$.

(4) If p is any nonzero projection of B, then $pBp \simeq B$.

(The fact that B satisfies (1)–(4) follows easily from the results of [**2**, §3].)

If σ is a nonzero endomorphism of B, define $A(\sigma)$ to be the C^*-algebra of continuous functions f: $[0, 1] \to B$ such that $f(1) = \sigma(f(0))$. $\text{Prim}(A(\sigma)) = \{J_t\colon 0 \leqslant t < 1\}$, where $J_t = \{f \in A(\sigma)\colon f(t) = 0\}$, and $\text{Prim}(A(\sigma))$ is homeomorphic to a circle under the identification $J_t \leftrightarrow e^{2\pi i t}$.

PROPOSITION 1.1. *$A(\sigma)$ is projectionless if* (*and only if*) $\sigma(1) \neq 1$.

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PROOF. $\tau \circ \sigma$ is a trace on B, so $\tau \circ \sigma = \lambda\tau$ for some λ, $0 < \lambda \leq 1$, and $\lambda = 1$ if and only if $\sigma(1) = 1$. If f is a projection of $A(\sigma)$, then $f(t)$ is a projection of B for each t. If t_1 and t_2 are sufficiently close, then $\|f(t_1) - f(t_2)\| < 1$, so $f(t_1) \sim f(t_2)$ by [**11**, Lemma 1.8], and thus $\tau(f(t_1)) = \tau(f(t_2))$. Therefore $\tau \circ f$: $[0, 1] \to [0, 1]$ is continuous and locally constant, hence constant. But $\tau \circ f(1) = \lambda(\tau \circ f(0))$, so either $\lambda = 1$ or $\tau \circ f \equiv 0$. (Alternate proof that $\tau \circ f$ is constant: it can take only algebraic values.)

The algebras A_n will be $A(\sigma_n)$ for appropriately chosen nonunital endomorphisms σ_n. The idea is the following. Let $A(\sigma_n)$ be given, and suppose $\sigma_n(1) = p$ with $0 < \lambda = \tau(p) < 1$. Set $\mu = \lambda^{1/2}/(1 + \lambda^{1/2})$, and let q, r be projections of B with $\tau(q) = \mu$, $\tau(r) = \lambda(1 - \mu)$, and $q \perp r$. Set $s = 1 - q - r$. Then $B \simeq qBq$ and $B \simeq (1 - q)B(1 - q)$, and the second isomorphism can be chosen to identify p with r. With these identifications, we can define ϕ: $A_n \to C([0, 1], B)$, as follows:

$$[\phi(f)](t) = \begin{bmatrix} f(t/2) & 0 & 0 \\ 0 & f((t+1)/2) & \\ 0 & & \end{bmatrix}$$

where elements of B are written symbolically as a 3×3 matrix:

$$x \leftrightarrow \begin{bmatrix} qxq & qxr & qxs \\ rxq & rxr & rxs \\ sxq & sxr & sxs \end{bmatrix},$$

$$[\phi(f)](0) = \begin{bmatrix} f(0) & 0 & 0 \\ 0 & f(1/2) & \\ 0 & & \end{bmatrix}, \quad [\phi(f)](1) = \begin{bmatrix} f(1/2) & 0 & 0 \\ 0 & f(0) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now $B \simeq (q + r)B(q + r)$, and the isomorphism can be chosen to identify q with r, since $\lambda(1 - \mu)/\mu = \mu/(1 - \mu)$. Therefore, there should be an isomorphism σ_{n+1} of B onto $(q + r)B(q + r)$ such that $\sigma_{n+1}([\phi(f)](0)) = [\phi(f)](1)$.

There are some technical problems with this approach, so the actual construction uses a slightly modified approach.

2. Automorphisms of AF algebras. In this section we obtain a result about automorphisms of certain simple AF algebras, which is perhaps of independent interest, and which will be used in the construction.

LEMMA 2.1. *Let D be an AF algebra, C a finite-dimensional C^*-subalgebra of D. Then the commutant R of C in D is an AF algebra. In fact, if $D = [\bigcup D_n]^-$ with each D_n finite-dimensional and $C = D_1$, then $R = [\bigcup R_n]^-$, where R_n is the commutant of C in D_n.*

PROOF. It may be assumed without loss of generality that D is unital and C is a unital subalgebra. If $p_1, \ldots, p_n$ are the minimal central projections of C, then every element of D can be written as an $n \times n$ matrix (x_{ij}), where $x_{ij} \in p_iDp_j$. The elements of C are "diagonal" matrices, and a routine argument shows that any element of R must also be diagonal, with the ith block in the commutant of p_iCp_i in p_iDp_i; and any such matrix defines an element of R. Thus $R \simeq \bigoplus_{i=1}^{n} p_iRp_i$, and so by restricting to p_iDp_i, it suffices to assume that C is a full matrix algebra in D. But then $D \simeq C \otimes R$ in standard fashion, and the result follows.

The next lemma is a slight variant of [3, Lemma 2.4] and [9, Theorem 3.8].

LEMMA 2.2. *Let D be a simple unital AF algebra with $K_0(D)$ totally ordered; and let α be an automorphism of D, and let C be a finite-dimensional C^*-subalgebra of D. Then there is a unitary $u \in D$ with $\alpha(c) = ucu^*$ for all $c \in C$.*

PROOF. The hypotheses imply that D has unique normalized trace τ and $p \sim q \Leftrightarrow \tau(p) = \tau(q)$. Thus $p \sim \alpha(p)$ for all projections $p \in D$. As in [3] let $e_{ij}^{(k)}$ be a set of matrix units for C, and $f_{ij}^{(k)} = \alpha(e_{ij}^{(k)})$. Then $e_{11}^{(k)} \sim f_{11}^{(k)}$ for each k via a partial isometry w_k. Let $u = \Sigma_k \Sigma_i f_{i1}^{(k)} w_k e_{1i}^{(k)}$.

THEOREM 2.3. *Let D be a simple unital AF algebra with $K_0(D)$ totally ordered. Then* Aut(D) *is path-connected in the topology of pointwise (norm-) convergence. In fact, if α_0 and α_1 are automorphisms of D, there is a norm-continuous path (u_t) $(0 < t < 1)$ of unitaries of D such that, if $\alpha_t =$ ad u_t, then (α_t), $0 \leqslant t \leqslant 1$, is a continuous path of automorphisms from α_0 to α_1.*

PROOF. Write $D = [\bigcup D_n]^-$ with D_n finite-dimensional and $D_n \subseteq D_{n+1}$. Let α be an automorphism of D. We will find a path from α to the identity automorphism. Let $u_1 = 1$, and for each $n > 1$, let u_n be a unitary of D with ad $u_n = \alpha$ on D_n. Connect u_n and u_{n+1} by a path as follows. $u_n^* u_{n+1}$ is in the commutant of D_n, which is an AF algebra and thus has a path-connected unitary group. For $n \leqslant t \leqslant n + 1$, define a continuous path $\{v_t\}$ of unitaries in the commutant of D_n with $v_n = 1$ and $v_{n+1} = u_n^* u_{n+1}$. Set $u_t = u_n v_t$. Then (u_t) $(n \leqslant t \leqslant n + 1)$ is a continuous path from u_n to u_{n+1}, and ad $u_t = \alpha$ on D_n. For $0 < t \leqslant 1$, define $\alpha_t =$ ad $u_{1/t}$. Thus, as $t \to 0$, $\alpha_t \to \alpha$ pointwise on $\bigcup D_n$, hence everywhere since $\|\alpha_t\| = 1$ for all t.

REMARK. The conclusions of 2.2 and 2.3 can be false if $K_0(D)$ is not totally ordered. For there exists a simple unital AF algebra D with exactly two normalized extremal traces, and an automorphism α of D which interchanges the two traces. Then α is not in the connected component of the identity in Aut(D) and cannot be unitarily implemented on every finite-dimensional subalgebra. Also, by [9, 3.8 and 3.9], if M is a simple unital AF algebra which is not UHF, then the inner automorphisms are not dense in Aut$(M^{\otimes}M)$.

3. Construction of A. Let B be the AF algebra defined in §1, p_1 a projection in B, $0 < \lambda_1 = \tau(p_1) < 1$, and σ_1 an isomorphism of B onto $p_1 B p_1$. Let $A_1 = A(\sigma_1)$. Inductively define A_n and σ_n: $A_n \to A_{n+1}$ as follows. Suppose $A_1, \ldots, A_n$ have been defined, with $A_n = A(\sigma_n)$, σ_n an isomorphism of B onto $p_n B p_n$, $0 < \lambda_n = \tau(p_n) < 1$. Let $p = p_n$, $\lambda = \lambda_n$. Let μ, q, r, s be as in §1. Choose a fixed isomorphism σ_{n+1} of B onto $(q + r)B(q + r)$ such that $\sigma_{n+1}(q) = r$. This is possible because $\tau(\sigma_{n+1}(q)) = [\mu + \lambda(1 - \mu)]\mu = \lambda(1 - \mu) = \tau(r)$. Set $A_{n+1} = A(\sigma_{n+1})$. σ_{n+1} induces isomorphisms α: $qBq \to rBr$ and β: $(1 - q)B(1 - q) \to qBq$ by restriction. Let γ: $B \to qBq$ and δ: $B \to (1 - q)B(1 - q)$ be arbitrary isomorphisms, with $\delta(p) = r$. This is possible since $\tau(\delta(p)) = \lambda(1 - \mu) = \tau(r)$.

Let θ_t be a pointwise-continuous path of automorphisms of qBq with $\theta_0 =$ identity and $\theta_1 = \beta \circ \delta \circ \gamma^{-1}$ (Theorem 2.3). Let w_t $(0 \leqslant t < 1)$ be a continuous path of unitaries in rBr with $w_0 = r$, such that $\operatorname{ad} w_t$ converges pointwise to $\alpha \circ \gamma \circ \sigma_n^{-1} \circ \delta^{-1}|rBr$ as $t \to 1$. Let $u_t = w_t + s$. Then u_t is unitary in $(1 - q) \cdot B(1 - q)$. Set $\pi_t = \operatorname{ad} u_t$ $(0 \leqslant t < 1)$.

Now define $\phi_n: A_n \to A_{n+1}$ as follows.

$$[\phi_n(f)](t) = \begin{bmatrix} (\theta_t \circ \gamma)[f(t/2)] & 0 & 0 \\ 0 & (\pi_t \circ \delta)[f((t+1)/2)] & \\ 0 & & \end{bmatrix} \quad \text{if } t < 1,$$

and

$$[\phi_n(f)](1) = \begin{bmatrix} (\beta \circ \delta)[f(1/2)] & 0 & 0 \\ 0 & (\alpha \circ \gamma)[f(0)] & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(with B represented as 3×3 matrices as in §1). Since

$$[\phi_n(f)](0) = \begin{bmatrix} \gamma[f(0)] & 0 & 0 \\ 0 & \delta[f(1/2)] & \\ 0 & & \end{bmatrix},$$

$\sigma_{n+1}([\phi_n(f)](0)) = [\phi_n(f)](1)$. It remains to prove that $\phi_n(f)$ is a continuous function on $[0, 1]$. The two diagonal blocks can be handled separately. In the lower block, since $r[\phi_n(f)](t)s$, $s[\phi_n(f)](t)r$, and $s[\phi_n(f)](t)s$ all approach zero as $t \to 1$, to prove continuity at 1 it suffices to prove that $r[\phi_n(f)](t)r \to (\alpha \circ \gamma)[f(0)]$ as $t \to 1$, i.e. $r[\phi_n(f)](t)r \to (\alpha \circ \gamma \circ \sigma_n^{-1})[f(1)]$. Continuity of both blocks everywhere therefore follows from the following lemma.

LEMMA 3.1. *Let D be a C^*-algebra, g: $[0, 1] \to D$ a continuous function, ω: $[0, 1] \to \operatorname{Aut}(D)$ a path which is continuous in the topology of pointwise convergence. Then h: $[0, 1] \to D$ defined by $h(t) = \omega_t(g(t))$ is continuous.*

PROOF. Let $\varepsilon > 0$, and $t_0 \in [0, 1]$. Let $\delta > 0$ be such that $\|g(t) - g(t_0)\| < \varepsilon/2$ and $\|\omega_t(g(t_0)) - \omega_{t_0}(g(t_0))\| < \varepsilon/2$ whenever $|t - t_0| < \delta$. Then $\|\omega(g(t)) - \omega(g(t_0))\| < \varepsilon/2$ for any automorphism ω.

$$\|h(t) - h(t_0)\| \leqslant \|\omega_t(g(t)) - \omega_t(g(t_0))\| + \|\omega_t(g(t_0)) - \omega_{t_0}(g(t_0))\| < \varepsilon$$

for $|t - t_0| < \delta$.

For the next step in the induction, set $p_{n+1} = q + r$, $\lambda_{n+1} = \mu + \lambda(1 - \mu)$.

We now let $A = \varinjlim \{A_n, \phi_n\}$.

LEMMA 3.2. *A is simple.*

PROOF. The closed ideals of A_n are in one-one correspondence with the closed subsets of the circle, under the identification in §1. If J is a proper closed ideal of A, set $J_n = J \cap A_n$. Fix n with $J_n \neq A_n$. For each k, $J_n = A_n \cap J_{n+k}$, and it follows that the nonempty closed set of T corresponding to J_n is invariant under rotation by angle $2\pi/2^k$. Since this is true for all k, the closed set is dense and therefore $J_n = \{0\}$. This is true for all n, so $J = \{0\}$ by [**1**, Lemma 4.5].

The fact that A is projectionless follows from the next proposition, which is well known (cf. [**7**, p. 9], [**8**, p. 81]). The proof is a routine exercise, and is omitted.

PROPOSITION 3.3. *If D is a C^*-algebra, the following are equivalent*:

(1) *D is projectionless.*

(2) *Every selfadjoint element of D has connected spectrum.*

(3) *There is a dense *-subalgebra D_0 of D, such that every selfadjoint element of D_0 has connected spectrum* (*in D*).

COROLLARY 3.4. *Let $D = \varinjlim \{D_\alpha, \psi_\alpha\}$. If each D_α is projectionless, then D is projectionless.*

PROPOSITION 3.5. *A is nuclear.*

PROOF. Each A_n is an extension of $C_0(R) \otimes B$ by B, and is therefore nuclear. Hence A is nuclear.

If K is the C^*-algebra of compact operators, then it follows from the argument of Proposition 1.1 that $A_n \otimes K$ is projectionless for each n. Therefore $A \otimes K$ is a simple stable projectionless C^*-algebra, and so any C^*-algebra Morita equivalent to A is nonunital and projectionless [**4**].

ADDED IN PROOF. The author has constructed a unital projectionless C^*-algebra using the methods of this paper [**15**].

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