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A REMARK ON THE GROWTH OF SOLUTIONS OF FIRST ORDER ALGEBRAIC DIFFERENTIAL EQUATIONS

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ABSTRACT. Two theorems on the growth of complex- and real-valued solutions of first order algebraic differential equations are proven.

1. Consider the algebraic differential equation

$$y'^2 = ay^3 + by^2 + cy + d$$

with certain complex constants a, b, c, d. It is known (see **[3]** and other references cited there) that given any comparison function this equation has complex-valued solutions whose absolute values grow more rapidly than the comparison function. In other words, there is no majorant expressed in terms of the coefficients of the equation for the solution of the differential equation on the x axis.

We consider the equation

$$F(x, y, y') = 0 \tag{1}$$

with real x and complex-valued (or real-valued) y, where F is a polynomial in respect to y and y'. Some estimates of the growth of solutions under corresponding conditions imposed on (1) were found in **[1]**, **[2]**.

We show below that for equations from a certain large subset E of equations of type (1) a majorant for their solutions on the x axis can be built.

2. In order to formulate the conditions defining the set E we rewrite (1) in the following way:

$$F_0\left(x, y, \frac{y'}{y}\right) \equiv \sum_{j=0}^{n} P_j\left(x, \frac{y'}{y}\right) y^{n-j} = 0 \tag{2}$$

with

$$P_j(x, u) = \sum_{k_j=0}^{m_j} F_{jk_j}(x) u^{m_j - k_j}, \qquad j = 0, 1, 2, 3, \dots, n. \tag{3}$$

DEFINITION. A differential equation (2) with complex-valued coefficients $F_{ml}(x)$ belongs to E if and only if:

(1) $m_0 \geqslant m_j$; $j = 1, 2, \dots, n$ (see (3)),

(2) $|F_{00}(x)| > 0$, $x \in [0, \infty)$, and

(3) $F_{kj}(x)$ are continuous functions on $[0, \infty)$ for all possible k and j.

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THEOREM 1. *Let $y(x)$ be a differentiable complex-valued solution of the differential equation* (2) *which belongs to E on the ray* $[x_0, \infty)$. *Then*

$$y(x) = O\left(\exp \int_{x_0}^{x} V(t)\, dt\right) \tag{4}$$

where

$$V(x) = \max_{\substack{0 < k_j < m_j \\ j=0,1,2,\ldots,n}} \left\{ \left| \frac{F_{jk_j}(x)}{F_{00}(x)} \right|^{1/k_j}, 1 \right\}. \tag{5}$$

We will also prove:

THEOREM 2. *Let*

$$F(x, y, y') \equiv \sum_{j=0}^{n} y^{n-j} \sum_{k_j=0}^{m_j} F_{jk_j}(x) y'^{m_j - k_j} = 0 \tag{6}$$

be given with continuous functions $F_{jk_j}(x)$ on $[0, \infty)$ *and*

$$|F_{00}(x)| \geqslant a_0 > 0, \qquad |F_{0m_0}(x)| \geqslant a_0 > 0, \qquad a_0 = \text{Const},$$
$$m_0 \geqslant m_j; \quad j = 1, 2, \ldots, n. \tag{7}$$

Then for each differentiable real-valued solution $y(x)$ on the ray $[x_0, \infty)$ one has

$$y(x) = O(V_0(x)) \tag{8}$$

where

$$V_0(x) = \operatorname*{Sup}_{\substack{0 < k_j < n_j \\ 0 < j < n}} |F_{jk_j}(x)|. \tag{9}$$

We will prove both of these theorems in the following sections of this paper.

3. Proof of Theorem 1. Because of the condition (2) in the definition of the set E we can put $F_{00}(x) \equiv 1$. Let $y(x)$ be a solution of (2) defined on $[x_0, \infty)$. We replace x by t according to the equality

$$t = \int_{x_0}^{x} V(x)\, dx \tag{10}$$

where $V(x)$ is defined in (5). Then $dy/dx = V(x) dy/dt$ and

$$F_0\left(x, y, \frac{y'}{y}\right) = \sum_{j=0}^{n} y^{n-j} \sum_{k_j=0}^{m_j} F_{jk_j}(x) V^{m_j - k_j}(x) \left(\frac{y'_t}{y}\right)^{m_j - k_j} = 0.$$

Dividing the last equation by $V^{m_0}(x) y^n$ we obtain

$$\sum_{k=0}^{m_0} F_{0k_0}(x) V^{-k_0}(x) \left(\frac{y'_t}{y}\right)^{n-k_0} = -\sum_{j=0}^{n-1} y^{-j} \sum_{k_j=0}^{m_j} F_{jk_j}(x) V^{m_j - m_0 - k_j}(x) \left(\frac{y'_t}{y}\right)^{m_j - k_j}. \tag{11}$$

Note now that

$$\frac{|F_{0k_0}(x)|}{V^{k_0}(x)} = \left(\sqrt[k_0]{\frac{|F_{0k_0}(x)|}{V(x)}}\right)^{k_0} \leqslant 1 \tag{12}$$

and

$$\frac{|F_{jk_j}(x)|}{V^{m_0-m_j+k_j}(x)} \leqslant \frac{|F_{jk_j}(x)|}{V^{k_j}(x)} \leqslant 1. \tag{13}$$

Thus the coefficients of (2.3) are bounded by 1.

4. *Case* 1. Suppose first that there is an infinite sequence of points $t_p \uparrow \infty$ such that:

(a) t_p is a local maximal point of the function

$$\varphi(t) = \ln|y(t)|/t \quad \text{and} \tag{14}$$

(b)

$$\overline{\lim_{t\to\infty}}\ \varphi(t) = \lim_{p\to\infty} \varphi(t_p) = \infty \tag{15}$$

(if $\overline{\lim}_{t\to\infty}\varphi(t) < \infty$ then Theorem 1 is correct).

Under these assumptions

$$\varphi'(t_p) = \frac{|y(t_p)|'}{|y(t_p)|}\cdot\frac{1}{t_p} - \frac{\ln|y(t_p)|}{t_p^2} = 0 \Rightarrow \frac{|y(t_p)|'}{|y(t_p)|} = \frac{\ln|y(t_p)|}{t_p} = \varphi(t_p) \tag{16}$$

and

$$\left|\frac{y'(t_p)}{y(t_p)}\right| \geqslant \frac{|y(t_p)|'}{|y(t_p)|} = \varphi(t_p) \underset{p\to\infty}{\to} \infty. \tag{17}$$

Since $|y'/y| > 1$ at t_p, it then follows from (11) and (17) that

$$\left|\frac{y'}{y}\right|^{m_0} - \sum_{j=1}^{m_0}\left|\frac{y'}{y}\right|^{m-j} \leqslant \left|\frac{y'}{y}\right|^{n}\sum_{j=1}^{n}\frac{1}{|y|^j} - \frac{n}{|y|}\left|\frac{y'}{y}\right|^{n}$$

and since $|y(t_p)| \underset{p\to\infty}{\to} \infty$ one has $(1 + o(1))|y'/y|^{m_0} \leqslant 0$. The last inequality is impossible. Hence in (15) one must have

$$\overline{\lim_{p\to\infty}}\ \varphi(t_p) < C < \infty \Rightarrow \overline{\lim_{t\to\infty}}\ \varphi(t) = \overline{\lim_{t\to\infty}}\ \frac{\ln|y(t)|}{t} < C$$

where C is a certain constant dependent on the solution $y(x)$ and satisfies

$$\ln|y(t)| < C't \Rightarrow \ln|y(x)| < C\int_{x_0}^{x} V(x)\,dx,$$

which completes the proof of Theorem 1 for Case 1.

Case 2. $\varphi(t)\uparrow\infty$, $t \geqslant t_0$, $t \to \infty$. In this case one has

$$\varphi'(t) = \frac{1}{t}\left\{\frac{|y(t)|'}{|y(t)|} - \frac{\ln|y(t)|}{t}\right\} \geqslant 0 \Rightarrow \frac{\ln|y(t)|}{t} \leqslant \frac{|y(t)|'}{|y(t)|} \leqslant \left|\frac{y'(t)}{y(t)}\right|$$

so that $|y'(t)/y(t)| \underset{t\to\infty}{\to} \infty$. Repeating the same considerations as in the last paragraph we conclude that $\varphi(t) < C$ which completes the proof of Theorem 1.

5. Proof of Theorem 2. The proof of this theorem is similar in its method to the proof of Theorem 1. We substitute now x for t from $t = \int_{x_0}^{x} V_0(x)\,dx$ with

$$V_0(x) = \sup_{\substack{0 \leqslant k_j \leqslant m_j \\ 0 \leqslant j \leqslant n}} |F_{jk_j}(x)|.$$

We assume $F_{00}(x) \equiv 1$ and get:

$$\sum_{j=0}^{n} \sum_{k_j=0}^{m_j} \frac{F_{jk_j}(x)}{V_0^{-m_j+m_0+k_j}(x)}\, y'^{m_j-k_j} y^j = 0 \tag{18}$$

where, for all j and k_j,

$$\left| \frac{F_{jk_j}(x)}{V_0^{m_0-m_j+k_j}(x)} \right| \leqslant 1. \tag{19}$$

(a) Suppose there is a sequence $t_p \uparrow \infty$ such that, for each p, $|y(t_p)|$ is a local maximal value and

$$\lim_{p\to\infty} |y(t_p)| = \overline{\lim_{t\to\infty}}\, |y(t)|. \tag{20}$$

Then $y'(t_p) = 0$ and from (18) and (19) it follows that

$$|y(t_p)|^n \leqslant C \sum_{k=0}^{n-1} |y^k(t_p)|$$

with a certain constant $C > 0$. If $|y(t_p)| \leqslant 1$ then $|y(x)| = O(1)$. Suppose instead that $|y(t_p)| \uparrow \infty$. Then for p large enough $|y(t_p)|^n \leqslant Cn|y(t_p)|^{n-1}$ and $(1 + o(1))|y(t_p)|^n \leqslant 0$. The last inequality is impossible. Thus in case (a) one has $y(x) = O(1)$.

(b) Suppose now $y(t) \uparrow \infty$, $t \geqslant t_0$, $t \to \infty$. If $|y'(t)| \leqslant C_0 < \infty$ then $|y(t)|' \leqslant C_0$. Hence $y(t) = O(t)$ and the theorem is proved. Assume therefore that there is a sequence $t_p \uparrow \infty$ with $y'(t_p) \uparrow \infty$. From (19) we have:

$$\sum_{k_0=0}^{m_0} \frac{F_{0k_0}(x)}{V^{m_0+k_0}(x)}\, y'^{m_0-k_0} = -\sum_{j=1}^{m} \frac{1}{y^j} \sum_{k_j=0}^{m_j} \frac{F_{jk_j}(x)}{V^{m_0-m_j+k_j}}\, y'^{m_j-k_j}$$

and for $t = t_p$ with p large enough one has

$$|y'(t_p)|^{m_0} - \sum_{j=1}^{m_0} |y'^{m_0-j}(t_p)| \leqslant \frac{1}{|y(t)|} |y'^{m_0}(t_p)| \Rightarrow (1 + o(1))|y'^{m_0}(t_p)| < 0.$$

But this inequality is impossible. Consequently $y'(t) \leqslant C$.

(c) If $y(t) \downarrow -\infty$ then by means of the transformation $y = -z$ we reduce this case to case (b), which completes the proof of Theorem 2.

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