PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 4, April 1980

UNIVERSALLY LUSIN-MEASURABLE AND BAIRE-1 PROJECTIONS

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ABSTRACT. It is obvious that a dual Banach space E^* is reflexive if and only if the natural projection P from E^{***} to E^* is weak* to weak continuous. In this paper it is proved that the next best condition on P, namely that P is weak* to weak universally Lusin-measurable is necessary and sufficient for E^* to have the Radon-Nikodým property. In addition we prove that if E is any Banach space that is complemented in its second dual by a weak* to weak Baire-1 projection, then E has the Radon-Nikodým property. We also prove that if E is a Banach space that is complemented in its second dual E^{**} by a projection P: $E^{**} \to E$ with $F = P^{-1}(0)$ weakly K-analytic; then saying that E^{**} has the Radon-Nikodým property is equivalent to saying that P is weak* to weak universally Lusin-measurable.

Let us fix some terminology and conventions. All topological spaces in this paper will be completely regular. The set of all Radon probability measures on a topological space (X, τ) will be denoted by $M^1_+(X, \tau)$.

DEFINITION 1. Let (X, τ_1) and (Y, τ_2) be two topological spaces and let f: $X \to Y$.

(i) Let $\mu \in M^1_+(X, \tau_1)$. The map f is μ-Lusin-measurable if for every compact set K in X and for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset K$ such that $\mu(K \setminus K_\varepsilon) < \varepsilon$ and the restriction of f to K_ε is continuous.

(ii) The map f is universally Lusin-measurable if f is μ-Lusin-measurable for every $\mu \in M^1_+(X, \tau_1)$.

DEFINITION 2. A topological space is K-analytic if it is the continuous image of a $K_{\sigma\delta}$ subset of a compact space.

It is clear that a complete separable metric space is K-analytic and if E is a Banach space, then $(E^*, \sigma(E^*, E))$ is K-analytic because it is a K_σ. Talagrand **[11]** showed that a weakly compactly generated (WCG) Banach space is K-analytic for its weak topology.

DEFINITION 3. Let E be a Banach space and W a subset of E. The set W is weakly K-analytic if $(W, \sigma(E, E^*))$ is K-analytic.

DEFINITION 4. Let (X, τ_1) and (Y, τ_2) be two topological spaces and let f: $X \to Y$. The map f is Baire-1 if there exists a sequence $(f_n)_{n \geq 1}$ of continuous functions from (X, τ_1) to (Y, τ_2) such that $f(x) = \lim_n f_n(x)$ for every $x \in X$.

DEFINITION 5. Let E be a Banach space and let (T, Σ, λ) be a probability space. A function f: $T \to E$ is Pettis-integrable if

(i) for every $x^* \in E^*$ the map $t \to \langle f(t), x^* \rangle$ is λ-measurable, and

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Presented to the Society, December 19, 1978; received by the editors February 8, 1979.

AMS (MOS) subject classifications (1970). Primary 46G10, 46B10, 28A30.

Key words and phrases. Radon-Nikodým property, universally Lusin-measurable maps.

(ii) for every A in Σ there exists x_A in E such that

$$\langle x_A, x^* \rangle = \int_A \langle f(t), x^* \rangle \, d\lambda$$

for every x^* in E^*. In this case we write $x_A = \text{Pettis-}\int_A f \, d\lambda$.

DEFINITION 6. Let E be a Banach space and let (T, Σ, λ) be a probability space. A function f: $T \to E$ is Bochner-integrable if there exists a sequence $(f_n)_{n \geq 1}$ of simple functions such that

(i) $\lim_n \|f(t) - f_n(t)\| = 0$ for λ-almost all $t \in T$, and

(ii) $\lim_n \int_T \|f(t) - f_n(t)\| d\lambda = 0$.

It is easy to see that one can define Bochner-$\int_A f \, dP = \lim_n \int_A f_n \, d\lambda$ for each A in Σ. This definition is independent of the choice of the sequence $(f_n)_{n \geq 1}$. For more details see [**2**, Chapter II].

DEFINITION 7. A Banach space E has the Radon-Nikodým (resp. weak Radon-Nikodým property) if for every probability space (T, Σ, λ) and every vector measure m: $\Sigma \to E$ such that $\|m(A)\| \leq \lambda(A)$ for every $A \in \Sigma$ there exists a Bochner-integrable (resp. Pettis-integrable) function f: $T \to E$ such that $m(A) = \text{Bochner-}\int_A f \, d\lambda$ (resp. $m(A) = \text{Pettis-}\int_A f \, d\lambda$) for every A in Σ.

Let (X, τ_1) and (Y, τ_2) be two topological spaces, let $\mu \in M_+^1(X, \tau_1)$ and let f: $X \to Y$ be a μ-measurable map. The measure $f(\mu)$ is defined on Borel sets of Y by $f(\mu)(B) = \mu(f^{-1}(B))$. It is easy to see that $f(\mu) \in M_+^1(Y, \tau_2)$.

The following theorem of P. A. Meyer [**9**, p. 126] is used in the sequel.

THEOREM 8. *Let τ_1 and τ_2 be two topologies on a set X. If τ_2 is finer then τ_1 and (X, τ_2) is K-analytic, then the identity map I: $(X, \tau_1) \to (X, \tau_2)$ is universally Lusin-measurable.*

Recall also [**9**, p. 162] that for any Banach space E the identity map I: $(E, \sigma(E, E^*)) \to (E, \| \; \|)$ is universally Lusin-measurable.

The following proposition is due to Odell and Rosenthal [**6**].

PROPOSITION 9. *Let E be a Banach space and let x^{**} be an element of E^{**} and suppose that x^{**} is Baire-1 when E^* is equipped with the w^*-topology. Then there exists a bounded sequence $(x_n)_{n \geq 1}$ of elements of E such x^{**} is the w^*-limit of $(x_n)_{n \geq 1}$.*

THEOREM 10. *Let E be a Banach space. The dual E^* of E has the Radon-Nikodým property if and only if the natural projection*

$$P: (E^{***}, \sigma(E^{***}, E^{**})) \to (E^*, \sigma(E^*, E^{**}))$$

is universally Lusin-measurable.

PROOF. If E^* has the Radon-Nikodým property then the identity map $(E^*, \sigma(E^*, E)) \to (E^*, \sigma(E^*, E^{**}))$ is universally Lusin-measurable ([**7**], [**8**], [**10**]). This implies that

$$P: (E^{***}, \sigma(E^{***}, E^{**})) \to (E^*, \sigma(E^*, E^{**}))$$

is universally Lusin-measurable since $(E^{***}, \sigma(E^{***}, E^{**})) \to (E^*, \sigma(E^*, E))$ is continuous. To prove the converse, consider the following diagram:

$$\begin{array}{ccc} (E^{***}, \sigma(E^{***}, E^{**})) & \overset{P_1}{\rightarrow} & (E^{*}, \sigma(E^{*}, E)) \\ & \searrow P & \downarrow I \\ & & (E^{*}, \sigma(E^{*}, E^{**})) \end{array}$$

where $I(x^*) = x^*$ for every x^* in E^* and $P(u) = P_1(u)$ for every u in E^{***}. It is enough to prove that I is universally Lusin-measurable [7]. Let $\lambda \in M_+^1(E^*, \sigma(E^*, E))$. By [**1**, p. 90], there is $\mu \in M_+^1(E^{***}, \sigma(E^{***}, E^{**}))$ such that $P_1(\mu) = \lambda$. Now P is μ-Lusin-measurable, therefore IP_1 is μ Lusin-measurable, and hence I is $P_1(\mu) = \lambda$ Lusin-measurable by [1, p. 88]. This completes the proof.

THEOREM 11. *Let E be a Banach space that is complemented in its bidual E^{**} by a projection P: $E^{**} \to E$, and suppose that $F = P^{-1}(0)$ is weakly K-analytic. Then the following statements are equivalent*:

(i) *The space E^{**} has the Radon-Nikodým property.*

(ii) *The projection P is universally Lusin-measurable from E^{**} with its w^*-topology to e with its weak topology.*

PROOF. To see that (i) implies (ii), consider the following diagram:

$$\begin{array}{ccc} (E^{**}, \sigma(E^{**}, E^{*})) & \overset{P}{\rightarrow} & (E, \sigma(E, E^{*})) \\ \downarrow I & \nearrow P_1 & \\ (E^{**}, \|\ \|) & & \end{array}$$

where $P_1(u) = P(u)$ and $I(u) = u$ for every u in E^{**}. Now (i) implies that I is universally Lusin-measurable [7] and thus $P = P_1 I$ is universally Lusin-measurable because P_1 is continuous.

To prove that (ii) implies (i), write $E^{**} = E \oplus F$ and let I: $E^{**} \to E^{**}$ be the identity map. This map can be written $I = P + Q$.

If we can prove that the identity map

$$I: (E^{**}, \sigma(E^{**}, E^*)) \to (E^{**}, \|\ \|)$$

is universally Lusin-measurable we will have completed the proof [7]. Observe first that $Q = I - P$ is universally Lusin-measurable from $(E^{**}, \sigma(E^{**}, E^*))$ to $(E^{**}, \sigma(E^{**}, E^*))$. Hence Q is universally Lusin-measurable from $(E^{**}, \sigma(E^{**}, E^*))$ to $(F, \sigma(E^{**}, E^*))$. Note that the identity

$$J: (F, \sigma(E^{**}, E^*)) \to (F, \sigma(F, F^*))$$

is universally Lusin-measurable by Theorem 8. Therefore,

$$Q: (E^{**}, \sigma(E^{**}, E^*)) \to (F, \|\ \|)$$

is universally Lusin-measurable. In particular, Q: $(E^{**}, \sigma(E^{**}, E^*)) \to (E^{**}, \|\ \|)$ is universally Lusin-measurable. Hence

$$I = P + Q: (E^{**}, \sigma(E^{**}, E^*)) \to (E^{**}, \|\ \|)$$

is universally Lusin-measurable. This completes the proof.

Kuo [**5**] proved that if E is a Banach space such that E^{**}/E is separable then E^* and E^{**} have the Radon-Nikodým property. The following example shows that the assumption E^{**}/E is separable cannot be weakened to the assumption that E^{**}/E is weakly K-analytic (or even reflexive).

EXAMPLE. Banach spaces E such that E^{**}/E weakly K-analytic with E^{**}, E^* or E failing the Radon-Nikodým property.

Let $E = JT$ the James tree space [4]. It is known that all its even duals have the Radon-Nikodým property and all its odd duals fail the Radon-Nikodým property and for every $n \geq 0$ we have $E^{(n+2)} = E^{(n)} \oplus l_2(\Gamma)$ with Γ uncountable. Therefore $E^{(2n)}/E^{(2n-2)}$ is reflexive but E^{2n-1} fails the Radon-Nikodým property for every $n \geq 1$. Also $E^{(2n+1)}/E^{(2n-1)}$ is reflexive but $E^{(2n+1)}$ fails the Radon-Nikodým property for every $n \geq 1$. It is also worth noting that all the even duals of the James tree space satisfy the conditions of Theorem 11.

THEOREM 12. *Let E be a Banach space that is complemented in E^{**} by a projection P: $E^{**} \to E$.*

(i) *If for every x^* in E^* the map x^*P is Baire-*1 *when E^{**} is equipped with the w^*-topology, then E has the weak Radon-Nikodým property.*

(ii) *If P: $(E^{**}, \sigma(E^{**}, E^*)) \to (E, \sigma(E, E^*))$ is Baire-*1 *then E has the Radon-Nikodým property.*

PROOF. (i) Let (T, Σ, λ) be a probability space and m: $\Sigma \to E$ be a vector measure such that $\|m(A)\| \leq \lambda(A)$ for every A in Σ. By [3] there exists f: $T \to E^{**}$ such that x^*f is λ-measurable for every x^* in E^* and

$$\langle x^*, m(A)\rangle = \int_A \langle x^*, f(t)\rangle \, d\lambda$$

for every A in Σ. To complete the proof it is enough to show that Pf is Pettis-integrable and

$$m(A) = \text{Pettis-}\int Pf \, d\lambda$$

for every A in Σ. To this end, fix $x^* \in E^*$ and recall that x^*P is Baire-1 by hypothesis.

According to Proposition 9 there is a sequence $(x_n^*)_{n \geq 1}$ in E^* such that $\langle u, x^*P\rangle = \lim_n \langle x_n^*, u\rangle$ for every u in E^{**}. This implies that

$$x^*Pf(t) = \langle f(t), x^*P\rangle = \lim_n \langle x_n^*, f(t)\rangle$$

for every $t \in T$. Hence x^*Pf is λ-measurable. On the other hand we can write

$$\langle m(A), x^*\rangle = \langle m(A), x^*P\rangle = \lim_n \langle x_n^*, m(A)\rangle = \lim_n \int_A \langle x_n^*, f(t)\rangle \, d\lambda$$

$$= \int_A \lim_n \langle x_n^*, f(t)\rangle \, d\lambda = \int_A \langle f(t), x^*P\rangle \, d\lambda = \int_A \langle x^*, Pf(t)\rangle \, d\lambda$$

for every A in Σ. Since x^* was arbitrary, this implies that Pf is Pettis-integrable and $m(A) = \text{Pettis-}\int_A Pf \, d\lambda$ for every A in Σ.

To prove (ii), recall that as a consequence of a theorem found in [2, p. 88] a weakly compactly generated Banach space with the weak Radon-Nikodým property has the Radon-Nikodým property. Thus it suffices to prove that the hypothesis

(ii) guarantees that E is weakly compactly generated. To this end let $(P_n)_{n\geq 1}$ be a sequence of continuous functions from $(E^{**}, \sigma(E^{**}, E^*)) \to (E, \sigma(E, E^*))$ such that $P(u) = \text{weak-lim}_n P_n(u)$ for every u in E^{**}. Let K be the unit ball of E^{**}, and let $a_n = \sup_{x\in K}\|P_n(x)\|$ and set $b_n = \max(a_n, 1)$ for every $n \geq 1$.

It is easy to see that the set $C = \bigcup_{n=1}^{\infty}(1/nb_n)\, P_n(K)$ is weakly compact in E and the closed linear span of C is E. This shows that E is weakly compactly generated and finishes the proof.

EXAMPLE. A Banach space that is complemented in its second dual by a weak* to norm Baire-1 projection.

Let P be the natural projection from l_∞^* to l_1. To see that P is weak* to weak Baire-1, define for each n, $P_n\colon l_\infty^* \to l_1$ by

$$P_n(\lambda)(m) = \begin{cases} \lambda(m) & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases}$$

It is easily checked that P_n is weak* to norm continuous and that $\lim_n P_n\lambda = P(\lambda)$ in l_1 norm for all λ in l_∞^*.

The author would like to thank Professor J. J. Uhl, Jr. for getting him interested in the material treated in this paper and for his helpful suggestions.

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